1

Solution

(a) Note that

- f is continuous and increasing,
- $f(-14) = -3$ and $f(13) = 3$.

Hence the range is $[-3, 3]$

(b) If we solve the equation $(f \circ f^{-1})(x) = x$ for $y = f^{-1}(x)$, then we get

$$\begin{aligned} f\left(f^{-1}(x)\right) &= x \\ f\left(y\right) &= x \\ \sqrt[3]{2y+1} &= x \\ 2y+1 &= x^3 \\ 2y &= x^3 - 1 \\ y &= \frac{x^3-1}{2} \end{aligned}$$

Hence an expression for f^{-1} is $f^{-1}(x) = \dfrac{x^3-1}{2}$.

(c) The domain is $[-3, 3]$ and the range is $[-14, 13]$.

2

Solution

We have:

1. sum of the roots: $\alpha + \beta = k$;
2. product of the roots: $\alpha\beta = k - 1$.

Hence we get

$$\begin{aligned} (\alpha+\beta)^2 &= k^2 \\ \alpha^2 + 2\alpha\beta + \beta^2 &= k^2 \\ \alpha^2 + \beta^2 &= k^2 - 2\alpha\beta \\ 17 &= k^2 - 2(k-1) \\ k^2 - 2k - 15 &= 0 \\ (k+3)(k-5) &= 0 \\ k &= -3, 5. \end{aligned}$$

3

Solution

Since $(x-4)$ is a factor of $f(x)$, we have

$$\begin{aligned} f(4) &= 0 \\ 4^3 - 2 \cdot 4^2 + 4a + b &= 0 \\ 32 + 4a + b &= 0 \\ 4a + b &= -32. \end{aligned}$$

Since division of $f(x)$ by $(x+2)$ leaves a remainder of 18, we have

$$\begin{aligned} f(-2) &= 18 \\ (-2)^3 - 2 \cdot (-2)^2 - 2a + b &= 18 \\ -16 - 2a + b &= 18 \\ -2a + b &= 34. \end{aligned}$$

Solving the system of two equations in a, b unknowns given above, we get $a = -11$ and $b = 12$.

4

Solution

(a) (i) The sum of the roots is given by

$$-\frac{a_4}{a_5} = -\frac{(-2)}{\frac{1}{4}} = 8.$$

(ii) The product of the roots is given by

$$(-1)^5 \cdot \frac{a_0}{a_5} = -\frac{(-128)}{\frac{1}{4}} = 512.$$

(b) From $q(x) = \frac{1}{4}(2x-2)^5 - 2(2x-2)^4 - 5(2x-2)^3 + 40(2x-2)^2 + 16(2x-2) - 128$, we find

$$a_5 = \frac{2^5}{4} = 8,$$
$$a_4 = -\frac{1}{4} \cdot 5 \cdot 2^4 \cdot 2 - 2 \cdot 2^4 = -72,$$
$$a_0 = -\frac{1}{4} \cdot 2^5 - 2 \cdot 2^4 + 5 \cdot 2^3 + 40 \cdot 2^2 - 16 \cdot 2 - 128 = 0.$$

(i) The sum of the roots is given by

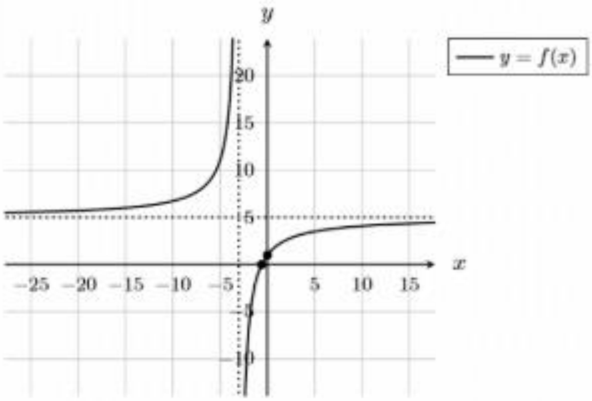
$$-\frac{a_4}{a_5} = -\frac{(-72)}{8} = 9.$$

(ii) The product of the roots is given by

$$(-1)^5 \cdot \frac{a_0}{a_5} = -\frac{0}{8} = 0.$$

Solution

(a) The sketch of the graph of $y = f(x)$ is given below:



The curve has

- x-intercept at $\left(-\frac{3}{5}, 0\right)$ and y-intercept at $(0, 1)$,
- vertical asymptote $x = -3$ and horizontal asymptote $y = 5$.

(b) If we solve the equation $(f \circ f^{-1})(x) = x$ for $y = f^{-1}(x)$, then we get

$$\begin{aligned}
f\left(f^{-1}(x)\right) &= x \\
f(y) &= x \\
1 + \frac{4y}{y+3} &= x \\
\frac{4y}{y+3} &= x - 1 \\
4y &= (x-1)(y+3) \\
4y &= (x-1)y + 3x - 3 \\
4y - (x-1)y &= 3x - 3 \\
(5-x)y &= 3x - 3 \\
y &= \frac{3x-3}{5-x}
\end{aligned}$$

Hence an expression for f^{-1} is $f^{-1}(x) = \dfrac{3x-3}{5-x}$.

(c) If we solve the equation $f(x) = f^{-1}(x)$ for x, then we get

$$\begin{aligned}
f(x) &= f^{-1}(x) \\
1 + \frac{4x}{x+3} &= \frac{3x-3}{5-x} \\
\frac{5x+3}{x+3} &= \frac{3x-3}{5-x} \\
(5x+3)(5-x) &= (3x-3)(x+3) \\
25x - 5x^2 + 15 - 3x &= 3x^2 + 9x - 3x - 9 \\
-5x^2 + 22x + 15 &= 3x^2 + 6x - 9 \\
-8x^2 + 16x + 24 &= 0 \\
x^2 - 2x - 3 &= 0 \\
(x+1)(x-3) &= 0 \\
x &= -1, 3.
\end{aligned}$$

(d) $|f(x)| = 2 \iff f(x) = \pm 2$. If we solve the equation $f(x) = -2$, then we get

$$\begin{aligned}
1 + \frac{4x}{x+3} &= -2 \\
\frac{4x}{x+3} &= -3 \\
4x &= -3x - 9 \\
7x &= -9 \\
x &= -\frac{9}{7}
\end{aligned}$$

If we solve the equation $f(x) = 2$, then we get

$$\begin{aligned}
1 + \frac{4x}{x+3} &= 2 \\
\frac{4x}{x+3} &= 1 \\
4x &= x + 3 \\
3x &= 3 \\
x &= 1
\end{aligned}$$

Hence the solution set is $\left(-\frac{9}{7}, 1\right)$.

(e) If we solve the inequality $f(|x|) < 2$, then we get

$$\begin{aligned}
f(|x|) &< 2 \\
1 + \frac{4|x|}{|x|+3} &< 2 \\
\frac{4|x|}{|x|+3} &< 1 \\
4|x| &< |x| + 3 \\
3|x| &< 3 \\
|x| &< 1
\end{aligned}$$

Hence the solution set is $(-1, 1)$.