COMPLEX NUMBERS:

The quadratic equation $x^2+1=0$ has no real solution for there is no real number whose square is -1. However, the equation $x^2=-1$ does have solutions, $\pm\sqrt{-1}=\pm i$, where $i=\sqrt{-1}$.

$\therefore i=\sqrt{-1} \Rightarrow i^2=-1$

A complex number is a number of the form $z=a+bi$, where $a, b\in\mathbb{R}$.

a is called the **real part** of z: $a=\text{Re}\,z \Rightarrow \text{Re}(a+bi)=a$

bi is called the imaginary part of z: $b=\text{Im}\,z \Rightarrow \text{Im}(a+bi)=b$ Some books may define $\text{Im}\,z$ as bi

A set of complex numbers is represented by $\mathbb{C}=\{a+bi: a,b\in\mathbb{R}\}$

Exercise:

1 Simplify: $i^3, i^4, i^5, i^6, i^{29}$

2 Simplify: $i^{4n}, i^{4n+1}, i^{4n+2}, i^{4n+3}$, where n is an integer.

3 Simplify: $\frac{1}{i}, \frac{1}{i^2}, \frac{1}{i^3}, \frac{1}{i^{15}}$

Operations with complex numbers:

When you add, subtract, multiply and divide two complex numbers $a+bi$ and $c+di$, the result is another complex number.

Exercise: Write the following complex numbers in the form $a+bi$.

(i) $(3+7i)+(2+5i)$ **(ii)** $(2\sqrt{3}-\sqrt{5}i)-(-\sqrt{3}-3\sqrt{5}i)$ **(iii)** $(p+qi)-(r+si)$

5 +12i

(iv) $(2+3i)(3+i)$ **(v)** $(1+\sqrt{3}i)(1-\sqrt{3}i)$ **(vi)** $(3+i)^3$ **(vii)** $\frac{2+3i}{3+i}$ **(viii)** $\frac{\sqrt{3}+i}{\sqrt{3}-i}$

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6+2i+9i+3(i)²

(ix) $\frac{1}{3+4i}$ **(x)** $\frac{2+3i}{-2i}$.

p+qi−r−si

p−r + i(q−s)

Definition: A **complex conjugate** of a complex number $z = a + bi$ is the complex number $\bar{z} = a - bi$. $\bar{z}$ is also denoted by z^*

$z = a + bi$ is a complex number. Show that $z + \bar{z}$ and $z\bar{z}$ are both real numbers. What about $z - \bar{z}$?

With complex numbers we cannot use the inequality symbols $>$ and $<$. One of the rules of inequality is that, if $a > b$ and $c > 0$, then $ac > bc$. So if $a > 0$ and $a > 0$, then $a \times a > 0 \times a \Rightarrow a^2 > 0$. What about i ? Is $i > 0$ or $i < 0$?

Let us assume $i > 0$ and do some usual algebra on it. Multiplying both sides by $i > 0$, we will have $\quad i \times i > 0 \times i \Rightarrow i^2 > 0 \Rightarrow -1 > 0$

If we assume that $i < 0$, then adding $-i$ to both sides will give

$i < 0 \Leftrightarrow i + (-i) < 0 + (-i) \Leftrightarrow 0 < -i \Rightarrow -i > 0 \Rightarrow (-i)^2 > 0 \Rightarrow -1 > 0$

Hence both premises $i > 0$ and $i < 0$ will lead to the contradiction $-1 > 0$. **Hence we need to make the rule that the relations $>$ and $<$ cannot be used to compare complex numbers.**

If a complex number $a + bi = 0$, then it follows that $a = -bi \Leftrightarrow a^2 = (-bi)^2 \Leftrightarrow a^2 = -b^2$. Now $a^2 > 0$ and $-b^2 < 0$. Hence $a^2 = -b^2$ only if $a = 0$ and $b = 0$. So if a complex number is zero, both its real part and complex part should be zero.

Combining this with the rule for subtraction shows that,

$(a + bi) = (c + di) \Leftrightarrow (a + bi) - (c + di) = 0 \Leftrightarrow (a - c) + (b - d)i = 0 \Leftrightarrow a - c = 0$ and $b - d = 0$
$\Leftrightarrow a = c$ and $b = d$.

Exercise: 1 Simplify: **(i)** $\dfrac{(3+4i)(2-i)}{3-i}$ **(ii)** $\dfrac{1+5i}{(1+i)(3-2i)}$ **(iii)** $\dfrac{(2+i)^3}{(1-i)^2}$ **(iv)** $\dfrac{1}{2-i} - \dfrac{1}{2+i}$

(v) $\dfrac{1}{x+3i} + \dfrac{1}{x-iy}$ **(vi)** $\dfrac{1}{(1+i)^2} - \dfrac{1}{(1-i)^2}$

2 Find z when: **(i)** $z(2+3i) = 4+i$ **(ii)** $(z+1)(2-i) = 3-4i$ **(iii)** $\dfrac{1}{z} = \dfrac{1}{3+4i} + \dfrac{2}{5+5i}$

(iv) $\frac{z}{1+2i}+\frac{z-1}{5i}=\frac{-2+i}{1-2i}$

Exercise: 2

1 The complex numbers *z* and *w* satisfy the simultaneous equations

$$(1+i)z+2w=3+7i$$
$$3z-(1+i)w=7+20i$$

Find *z* and *w*, giving your answer in the form $a+bi$.

2 (i) (a) Evaluate $(1+i)^2$, where $i=\sqrt{-1}$.

(b) Prove by mathematical induction that $(1+i)^{4n}=(-4)^n$ where $n\in N^*$.

(c) Hence or otherwise, find $(1+i)^{32}$.

3 Express the complex number

$z=\frac{(1+5i)(2-3i)}{(4+i)(1+2i)}$ in the form $a+bi$ and the complex number

4 The complex numbers w and z are such that $w=\frac{az+b}{z+c}$, where a,b and $c\in\mathbb{R}$.

Given that $w=3i$ when $z=-3i$, and that $w=1-4i$ when $z=1+4i$, show that $b=9$ and find the values of *a* and *c*.

5 Express the square roots of the complex number $-7-24i$ in the form $a+bi$ where a and b are real numbers.

6 Find the square roots of the imaginary number $2i$ in the form $x+yi$, where *x* and *y* are real.

7 The complex number *z* is given by $z=1+\frac{i}{i-\sqrt{3}}$.

Express *z* in the form $a+bi$, giving the exact values of the real constants *a* and *b*.

8 The complex number z satisfies the equation

$\sqrt{z}=\frac{2}{1-i}+1-4i$. Express z in the form $x+yi$ where $x,y\in\mathbb{Z}$.

9 Consider the equation $2(p+iq)=q-ip-2(1-i)$, where p and q are both real numbers. Find p and q.

10 The complex number z satisfies $i(z+2)=1-2z$, where $i=\sqrt{-1}$. Write z in the form $z=a+ib$ where a and b are real numbers.

Complex roots of polynomial equations:

If a quadratic equation with real coefficients has complex roots, these roots are conjugates.

Proof: Suppose the equation $ax^2+bx+c=0$ has complex roots then, $b^2-4ac<0$, so

$\sqrt{b^2-4ac}=i\sqrt{4ac-b^2}$ Then $ax^2+bx+c=0 \Rightarrow x=-\dfrac{b}{2a}\pm\dfrac{\sqrt{b^2-4ac}}{2a}=-\dfrac{b}{2a}\pm\dfrac{i\sqrt{4ac-b^2}}{2a}$.

Let $-\dfrac{b}{2a}=\alpha$ and $\beta=\dfrac{\sqrt{4ac-b^2}}{2a}$, then $x=\alpha\pm\beta i$. The roots are conjugates.

Example: If one of the roots of the equation $z^2+pz+q=0$ is $3-2i$, find p and q.

If $3-2i$ is one root, then the other root is its conjugate, $3+2i$.

$\therefore$ sum of roots $=3+2i+3-2i=6$, and the product of the roots is

$(3+2i)(3-2i)=3^2+2^2=13$

The equation is $z^2-(\text{sum of roots})\,z+\text{product of roots}=0 \Rightarrow z^2-6z+13=0$

$\therefore p=-6,\ q=13$

Complex roots of any polynomial with real coefficient, will always occur in conjugate pairs. Hence if a cubic equation has complex roots, one root has to be real and the others conjugates.

Example: Given that $1-i$ is a root of the equation $z^3-5z^2+8z+p=0$, find the other two roots and the value of p.

$1+i$ is the other root. The quadratic equation that has $1-i$ and $1+i$ as roots is $z^2-2z+2=0$. To find the third root, let us divide,

	z	-3			
z^2-2z+2	z^3	$-5z^2$	$+8z$	$+p$	
	z^3	$-2z^2$	$+2z$		
		$-3z^2$	$+6z$	$+p$	
		$-3z^2$	$-6z$	-6	
				p	$+6$

Since z^2-2z+2 is a factor of $f(z)$, $p+6=0 \Rightarrow p=-6$ and the root is $z=3$.

Exercise: 3

1 If one of the roots of the equation $z^2+pz+q=0$ is $2+5i$, find p and q.

2 If one of the roots of the equation $z^2+pz+q=0$ is $1-\sqrt{3}i$, find p and q.

3 Find the quadratic equation with roots (i) $2i, -2i$ (ii) $-2+\sqrt{3}i, -2-\sqrt{3}i$

4 For the following cubic equations, each with a given complex root, find the other two roots and the value of p:

(i) $z^3-4z^2+6z+p=0; 1+i$ **(ii)** $z^3-3z^2+9z+p=0; 2+3i$

(iii) $2z^3+z^2+4z+p=0; -1+2i$

5 Show that $2+i$ is a root of the equation $2z^3-9z^2+14z-5=0$ and find the other two roots.

6 (a) Suppose that the cubic equation $ax^3+bx^2+cx+d=0$ has roots α, β, γ, using the factor theorem $ax^3+bx^2+cx+d=a(x-\alpha)(x-\beta)(x-\gamma)$, by equating coefficients show that

(i) $\alpha+\beta+\gamma=-\frac{b}{a}$ **(ii)** $\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$ **(iii)** $\alpha\beta\gamma=-\frac{d}{a}$

(b) Given that the equation $z^3-z^2+3z+p=0$ has a root $1+2i$ use (i) and (iii), to find the other roots and the value of p. Check that (ii) holds for this equation.

7 Given that $1+i$ is a root of the equation $z^3+pz^2+qz+6=0$, find the other two roots and the values of p and q.

8 For the following cubic equations, each with the given root, find the other two roots and the values of p and q.

(i) $z^3+3z^2+pz+q=0; -2-i$ **(ii)** $2z^3-5z^2+pz+q=0; 1+3i$

(iii) $z^3+pz^2+qz-25=0; 4-3i$

9 (i) Find the real root of the equation $z^3-1=0$. Hence find the two complex roots. (Notice that the roots are the three cube roots of 1.)

(ii) If either of the complex root is denoted by ω, show that the other complex root is ω^2.

(iii) Show that ω and ω^2 are conjugates.

(iv) Show that $1+\omega+\omega^2=0$

10 (i) Find the three cube roots of -1. (one real and two complex)

(ii) If either of the complex roots is denoted by λ, show that the complex root is $-\lambda^2$.

(iii) Show that λ and $-\lambda^2$ are conjugates.

(iv) Show that $1+\lambda^2=\lambda$

11 Find the cubic equations with roots: **(i)** $1+i, 1-i, 3$ **(ii)** $-1+\sqrt{3}i, -1-\sqrt{3}i, 2$

12 Find the other three roots of the quartic equation with the given root:

(i) $z^4-4z^3+3z^2+2z-6=0; 1-i$ **(ii)** $z^4-6z^3+23z^2-34z+26-0; 2+3i$

13 Find the integer root of the equation $z^5-z^4+4z^3-2z^2+8=0$. Given that $1+i$ is also a root of this equation, find the other three roots.

The Argand diagram:

Just as we can represent real numbers on a real number line, we can represent complex numbers in a plane. The complex number $z=x+iy$ can be represented by the point $P(x, y)$, where all the points $(x, 0)$ on the x-axis represent real numbers and all the points $(0, y)$ on

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1 (a) Write down the modulus and argument of the complex number -64.

(b) Hence solve the equation $z^4 + 64 = 0$, giving your answers in the form $r(\cos\vartheta + i\sin\vartheta)$, where $r > 0$ and $-\pi < \vartheta \leq \pi$.

(c) Express each of these four roots in the form $a + ib$ and show, with the aid of a diagram, that the points in the complex plane which represent them form the vertices of a square.

2 (a) Express each of the complex numbers $1+i$ and $\sqrt{3}-i$ in the form $r(\cos\vartheta + i\sin\vartheta)$,where $r > 0$ and $-\pi < \vartheta \leq \pi$.

(b) Using your answers to part (a),

(i) show that $\dfrac{\left(\sqrt{3}-i\right)^5}{(1+i)^{10}} = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$

(ii) solve the equation $z^3 = (1+i)\left(\sqrt{3}-i\right)$ giving your answers in the form $a+ib$, where a and b are real numbers to be determined to two decimal places.

3 (a) By considering $z = \cos\vartheta + i\sin\vartheta$ and using de Moivre's theorem, show that

$$\sin 5\vartheta \equiv \sin\vartheta\left(16\sin^4\vartheta - 20\sin^2\vartheta + 5\right).$$

(b) Find the exact values of the solutions of the equation $16x^4 - 20x^2 + 5 = 0$.

(c) Deduce the exact values of $\sin\left(\dfrac{\pi}{5}\right)$ and $\sin\left(\dfrac{2\pi}{5}\right)$, explaining clearly the reasons for your answers.

4 (a) Show that the non-real cube roots of unity satisfy the equation $z^2 + z + 1 = 0$.

(b) The real number a satisfies the equation $\frac{1}{a-\omega+\omega^2}+\frac{1}{a+\omega-\omega^2}=\frac{1}{2}$ where ω is one of the non-real cube roots of unity. Find the possible values of a.

5 **(a)** Verify that $z_1=1+e^{\frac{i\pi}{5}}$ is a root of the equation $(z-1)^5=-1$.

(b) Find the other four roots of the equation.

(c) Mark on an Argand diagram the points corresponding to the five roots of the equation.

Show that these roots lie on a circle, and state the centre and radius of the circle.

(d) By considering the Argand diagram, find

(i) arg z_1 in terms of π,

(ii) $|z_1|$ in the form $a\cos\left(\frac{\pi}{b}\right)$, where a and b are integers to be determined.

6 **(a)** **(i)** Show that $w=e^{\frac{2\pi i}{5}}$ is one of the fifth roots of unity.

(ii) Show that the other fifth roots of unity are 1, w^2, w^3 and w^4.

(b) Let $p=w+w^4$ and $q=w^2+w^3$, where $w=e^{\frac{2\pi i}{5}}$.

(i) Show that $p+q=-1$ and $pq=-1$.

(ii) Write down the quadratic equation, with integer coefficients, whose roots are p and q.

(iii) Express p and q as integer multiples of $\cos\left(\frac{2\pi}{5}\right)$ and $\cos\left(\frac{4\pi}{5}\right)$, respectively,

(iv) Hence obtain the values of $\cos\left(\frac{2\pi}{5}\right)$ and $\cos\left(\frac{4\pi}{5}\right)$ in surd form.

7 (a) (i) Use de Moivre's theorem to show that if $z = \cos\vartheta + i\sin\vartheta$, then $z^n + \frac{1}{z^n} = 2\cos n\vartheta$.

(ii) Write down the corresponding result for $z^n - \frac{1}{z^n}$.

(b) (i) Show that $\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^3 = A\left(z^6 - \frac{1}{z^6}\right) + B\left(z^2 - \frac{1}{z^2}\right)$ where A and B are numbers to be determined.

(ii) By substituting $z = \cos\vartheta + i\sin\vartheta$ in the above identity, deduce that

$$\cos^3\vartheta\sin^3\vartheta = \frac{1}{32}(3\sin 2\vartheta - \sin 6\vartheta).$$

8 (a) (i) Express $e^{\frac{i\vartheta}{2}} - e^{-\frac{i\vartheta}{2}}$ in terms of $\sin\left(\frac{\vartheta}{2}\right)$.

(ii) Hence, or otherwise, show that $\frac{1}{e^{i\vartheta}-1} = -\frac{1}{2} - \frac{i}{2}\cot\left(\frac{\vartheta}{2}\right)$, $e^{i\vartheta} \neq 1$.

(b) Derive expressions, in the form $e^{i\vartheta}$, where $-\pi < \vartheta \leq \pi$, for the four non-real roots of the equation $z^6 = 1$.

(c) The equation $\left(\frac{w+1}{w}\right)^6 = 1$ $(*)$ has one real root and four non-real roots.

(i) Explain why the equation has only five roots in all.

(ii) Find the real root.

(iii) Show that the non-real roots are $\frac{1}{z_1-1}, \frac{1}{z_2-1}, \frac{1}{z_3-1}, \frac{1}{z_4-1}$ where z_1, z_2, z_3 and z_4 are the non-real roots of the equation $z^6 = 1$.

(iv) Deduce that the points in an Argand diagram that represents the roots of equation (*) lie on a straight line.

9 (a) Express the complex number $2+2i$ in the form $re^{i\vartheta}$, where $r>0$ and $-\pi<\vartheta\leq\pi$.

(b) Show that one of the roots of the equation $z^3=2+2i$ is $\sqrt{2}e^{\frac{\pi i}{12}}$, and find the other two roots giving your answers in the form $re^{i\vartheta}$, where *r* is a surd and $-\pi<\vartheta\leq\pi$.

(c) Indicate on an Argand diagram points *A*, *B* and *C* corresponding to the three roots found in part **(b)**.

(d) Find the area of the triangle *ABC*, giving your answer in surd form.

(e) The point *P* lies on the circle through *A*, *B* and *C*. Denoting by w, α , β and γ the complex numbers represented by *P*, *A*, *B* and *C*, respectively, show that

$$\left|(w-\alpha)^2+(w-\beta)^2+(w-\gamma)^2\right|=6.$$

THE END☯☺