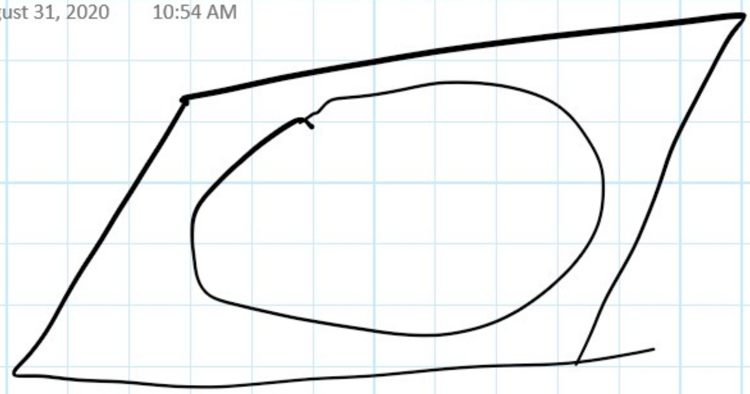
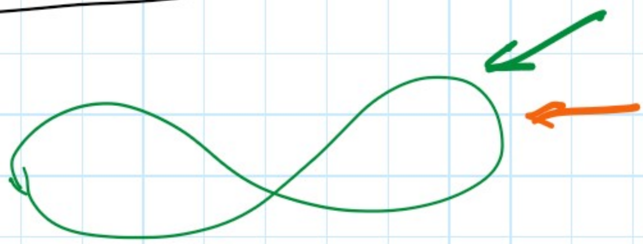
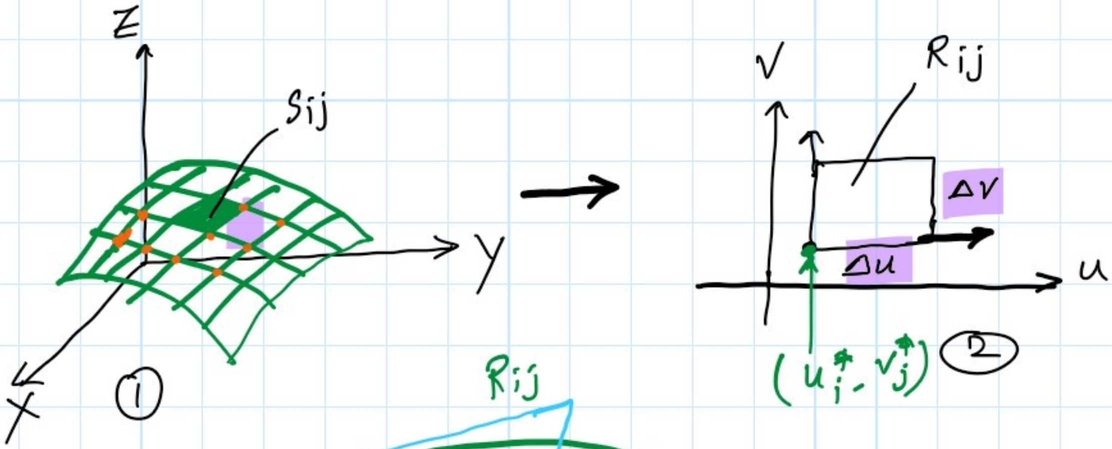
Surface integral, Parametric surfaces

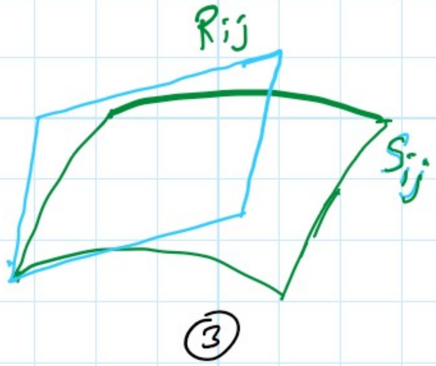
Monday, August 31, 2020 10:54 AM





$\iint_R$





$$\left| (\Delta u \, r_u^*) \times (\Delta v \, r_v^*) \right| = \left| r_u^* \times r_v^* \right| \Delta u \, \Delta v$$

$$\left| (\Delta u\, r_u^*) \times (\Delta v\, r_v^*) \right| = \left| r_u^* \times r_v^* \right| \Delta u \, \Delta v$$

So, an approximation to the area of S is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \| r_u^* \times r_v^* \| \Delta u \, \Delta v$$

$$\iint_D \| r_u \times r_v \| \, du\, dv$$

Definition:: If a smooth parametric surface S is given by the equation

$$r(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

$$(u,v) \in D.$$

and S is covered just once as (u,v) ranges throughout the parameter domain D, then the surface area of S is

$$A(S) = \iint_D \| r_u \times r_v \| \, dA$$

where $r_u = \dfrac{\partial x}{\partial u}\,\hat{i} + \dfrac{\partial y}{\partial u}\,\hat{j} + \dfrac{\partial z}{\partial u}\,k$,

& $r_v = \dfrac{\partial x}{\partial v}\,\hat{i} + \dfrac{\partial y}{\partial v}\,\hat{j} + \dfrac{\partial z}{\partial v}\,k$.

1.

$$\boxed{\iint_R f(x,y)\, dA}$$ is over a flat region R. & gives the mass of 'R' if $f(x,y)$ is the mass density function of 'R'.

What if 'R' is not flat?

Then the region is actually a surface 'S'.

A surface $\int$; $\iint_S f(x,y,z)\, ds$ ← surface area

This is mass density function.

The projection of our surface 'S' onto a co-ordinate plane gives us 'R'.

1.

a) $$\iint_S f(x,y,z)\, ds = \iint_R f(x,y,g(x,y)) \sqrt{g_x^2+g_y^2+1}\; dA$$

For surface $z = g(x,y)$

b) $$\iint_S f(x,y,z)\, ds = \iint_R f(x, g(x,z), z) \sqrt{g_x^2+g_z^2+1}\; dA$$

for surface $y = g(x,z)$

c) $$\iint_S f(x,y,z)\, ds = \iint_R f(g(y,z), y, z) \sqrt{g_y^2+g_z^2+1}\; dA$$

for surface $x = g(y,z)$.

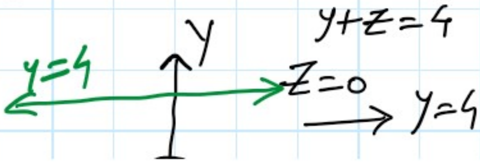
Ex $$\iint_S (x+2y+z)\, ds \quad ; \quad S \begin{cases} y+z=4 & \text{inside} \\ x^2+y^2=1 & \text{— cylinder.} \end{cases}$$

$z = 4-y$, $g_x = 0$, $g_y = -1$

$$\iint_R (x+2y+4-y)\sqrt{0^2+(-1)^2+1}\; dA.$$

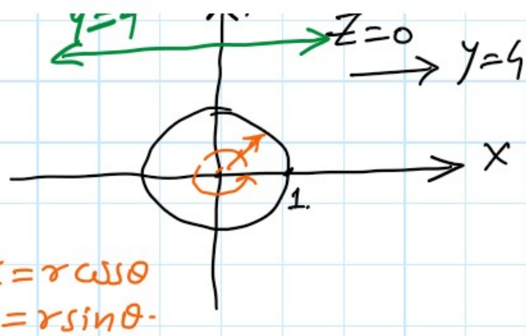
$x^2+y^2=1$

$y+z=4$



$$= \iint (x+y+4)\sqrt{2}\; dA.$$

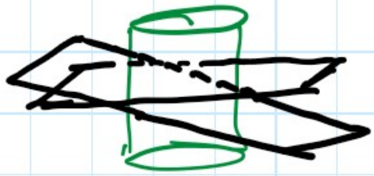
$$= \iint_R (x+y+4)\sqrt{2}\; dA.$$



$$= \int_0^{2\pi}\int_0^1 (r\cos\theta + r\sin\theta + 4)\sqrt{2}\; r\,dr\,d\theta$$

$x = r\cos\theta$
$y = r\sin\theta.$

$$= \boxed{4\sqrt{2}\,\pi}$$



Mass if $f(x,y,z) = x + 2y + z$ is mass density function.

parametric surface:-

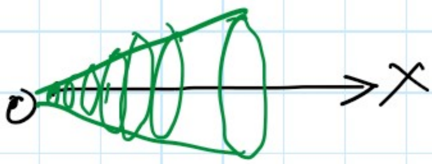
For $f(x,y,z)$ &
$$x = x(u,v)$$
$$y = y(u,v)$$
$$z = z(u,v)$$

$$\vec{r}(u,v) = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

$$f(x,y,z) = \vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

Ex: $X = \sqrt{y^2+z^2}$

$$\left\{\begin{array}{l} y = u,\quad z = v \\ x = \sqrt{u^2+v^2} \end{array}\right\}$$



For function already solved for a variable, let other variable be parameter.

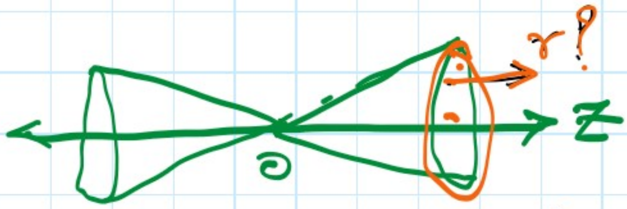
$$\vec{r}(u,v) = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

$$\vec{r}(u,v) = \sqrt{u^2+v^2}\,\hat{i} + u\,\hat{j} + v\hat{k}$$

$$\vec{r}(u,v) = \sqrt{u^2+v^2}\,\hat{i} + u\,\hat{j} + v\,\hat{k}$$

Ex $\quad x^2+y^2 = z^2$

$$z^2 = r^2$$
$$\Rightarrow z = r$$



$$x^2+y^2 = r^2$$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \begin{aligned} &= u\cos v \\ &= u\sin v \end{aligned}$$

$$z = r$$

r, θ
u, v

$$\vec{r}(u,v) = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$
$$= u\cos v\,\hat{i} + u\sin v\,\hat{j} + u\,\hat{k}$$

\# For the special case of a surface S with equation $z = f(x,y)$, where (x,y) lies in D and f has continuous partial derivative, we take u and v as parameters. The parametric eq^n are

$$x = u, \quad y = v, \quad z = f(u,v)$$

$$r_u = 1\,\hat{i} + 0\,\hat{j} + \frac{\partial f}{\partial u}\,\hat{k},$$

$$r_v = 0\,\hat{i} + 1\,\hat{j} + \frac{\partial f}{\partial v}\,\hat{k}$$

$$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial u} \\ 0 & 1 & \frac{\partial f}{\partial v} \end{vmatrix}$$

$$= -\frac{\partial f}{\partial u}\,\hat{i} - \frac{\partial f}{\partial v}\,\hat{j} + \hat{k}$$

$$\| r_u \times r_v \| = \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 + 1}$$

$$\|r_u \times r_v\| = \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 + 1}$$

Hence, surface area.

$$A(S) = \iint_D \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 + 1}\; dA.$$

Ex Find surface area of $x^2+y^2=z^2$.

Solⁿ $\vec{r}(u,v) = u\cos v\,\hat{i} + u\sin v\,\hat{j} + u\hat{k}$

$$\vec{r}_u = \cos v\,\hat{i} + \sin v\,\hat{j} + \hat{k}$$
$$\vec{r}_v = -u\sin v\,\hat{i} + u\cos v\,\hat{j} + 0\hat{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix}$$

$$= -u\cos v\,\hat{i} - u\sin v\,\hat{j} + u\hat{k}$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2}$$
$$= \sqrt{2u^2} = \sqrt{2}\,u.$$

$$\text{surface area} = \iint_D \|\vec{r}_u \times \vec{r}_v\|\; dA.$$

$$SA = \iint_D \sqrt{2}\,u\; dA.$$

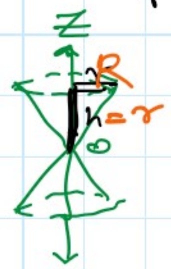
$r = u$
$\theta = v$

$$= \int_{v=0}^{2\pi}\int_{u=0}^{u=r} \sqrt{2}\,u\; du\,dv.$$

$$= \int_0^{2\pi} \frac{\sqrt{2}}{2} \left[u^2\right]_{u=0}^{u=r} dv$$

$$= \frac{\sqrt{2}}{2} \int_0^{2\pi} r^2 \, dv$$

$$= \frac{\sqrt{2}}{2} r^2 \left[v\right]_0^{2\pi} = \underline{\sqrt{2}\pi r^2}$$ — surface area of $x^2+y^2=z^2$



$z=h$

$S.A = \pi r\sqrt{r^2+h^2}$

verify:

$$S.A = \pi r \sqrt{r^2+h^2}$$
$$= \pi r \sqrt{r^2+r^2}$$
$$= \pi r \sqrt{2r^2}$$
$$= \pi r \sqrt{2}\, r$$
$$= \underline{\sqrt{2}\pi r^2}$$

Ex.

$$\iint_S \left(\frac{x-y}{\sqrt{2z+1}}\right) dS.$$

$S: \ \vec{r}(u,v) = (u+v)\,\hat{i} + (u-v)\,\hat{j} + (u^2+v^2)\,\hat{k}$

& $0 \le u \le 1, \ 0 \le v \le 2$

Surface area $= \iint_D \|\vec{r_u} \times \vec{r_v}\| \, dA$

✓ Parametric surface:

$$\iint_D f(\vec{r}(u,v))\, \|\vec{r_u} \times \vec{r_v}\| \, dA.$$

$\vec{r_u} = \hat{i} + \hat{j} + 2u\,\hat{k}$

$\vec{r_v} = \hat{i} - \hat{j} + 2v\,\hat{k}$

$$\vec{r}_u = \hat{i} + \hat{j} + 2u\hat{k}$$

$$\vec{r}_v = \hat{i} - \hat{j} + 2v\hat{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2u \\ 1 & -1 & 2v \end{vmatrix} = (2v+2u)\hat{i} - (2v-2u)\hat{j} + (-1-1)\hat{k}$$

$$= (2v+2u)\hat{i} + (2u-2v)\hat{j} - 2\hat{k}$$

$$x = u+v, \quad y = u-v, \quad z = u^2+v^2$$

$$\int_{v=0}^{v=2}\int_{u=0}^{u=1} \frac{u+v-u+v}{\sqrt{2(u^2+v^2)+1}} \cdot \sqrt{(2v+2u)^2+(2u-2v)^2+(-2)^2}\; du\, dv$$

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