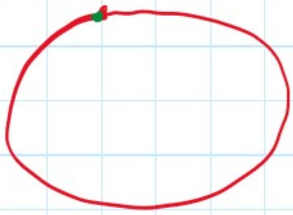
Green's Theorem

Sunday, August 23, 2020 3:53 PM



$$W = \int_C P\,dx + Q\,dy \qquad W = \int \underline{F} \cdot d\underline{\underline{r}}$$

$$F(x,y) = P\,\hat{i} + Q\,\hat{j}$$

$$\bar{r}(t) = x\,\hat{i} + y\,\hat{j}$$

$$r'(t) = dx\,\hat{i} + dy\,\hat{j}$$

$$\int F \cdot dr = \int F(x,y) \cdot r'(t) = \int P\,dx + Q\,dy$$

= work done.

$$= \oint_C P\,dx + Q\,dy$$

$\int_C P\,dx + Q\,dy$ has a curve C that encloses a region on a plane and 'C' is a simple closed curve that is travelled in the positive counter clockwise direction.

then the line $\int$ becomes $\oint_C P\,dx + Q\,dy$

Green's theorem:

A line $\int$ for a simple closed curve = $\iint$ over the region that curve contains.

$$\oint_C P\,dx + Q\,dy = \iint_R \left[\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dA.$$

R ← need to be closed curve.

$R \leftarrow$ need to be closed curve.

$\text{curl } F = 0$

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0.$$

→ If we get Integral = 0 then it is not necessary that the V-F is conservative.

for conservative $\Rightarrow \dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$

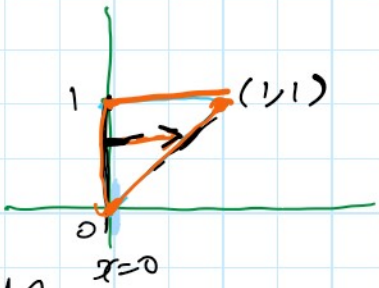
Green's thm deals with line Integral over simple closed curve through non-conservative vector field.

Ex

$$\oint_C x^3\,dx + xy\,dy \quad \text{for } C: \triangle\ (0,0),(1,1),(0,1)$$

$P = \underline{x^3}$, $Q = \underline{xy}$

$F(x,y) = x^3\,\hat{i} + xy\,\hat{j}$



$$\oint_C x^3\,dx + xy\,dy = \iint_R (y - 0)\,dA$$

$$= \int_{y=0}^{y=1} \int_{x=0}^{x=y} y \;dx\,dy$$

$W = \dfrac{1}{3}$

Ex

$$\oint_C 2x\,dx - \underline{3y}\,dy$$
P Q

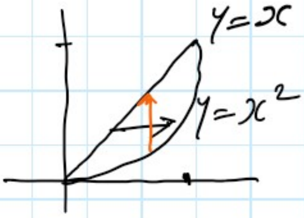
over a simple closed curve.

$$= \iint_R (\underline{0 - 0})\,dA$$

$= \underline{\underline{0}}$

<u>Ex</u> $\oint_C (x^2y + x^3)\,dx + 2xy\,dy$; C: path bounded by $y = x$ & $y = x^2$

$\qquad\quad P \qquad\qquad\qquad\quad Q$



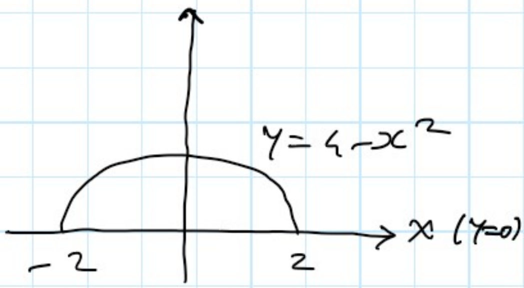
$$\iint_R (2y - x^2)\,dA$$

$$W = \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} 2y - x^2 \,dy\,dx$$

$$= \frac{1}{12}$$

<u>Ex</u> $W = \oint (y^2 + \cos x)\,dx + (x - \tan^{-1} y)\,dy$; C: $\boxed{y = 4 - x^2 \ \& \ y = 0}$

$\Rightarrow F(x,y) = (y^2 + \cos x)\,\hat{i} + \underline{(x - \tan^{-1} y)}\,\hat{j}$



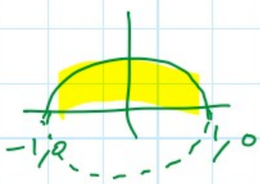
$$W = \iint_R (1 - 2y)\,dA$$

$$= \int_{x=-2}^{x=2} \int_{y=0}^{y=4-x^2} (1 - 2y)\,dy\,dx.$$

$$= \frac{-352}{15}$$

<u>Ex</u> $\oint x^2y\,dx + y^3\,dy$, C: $(-1,0) \rightarrow (1,0)$ & $x^2 + y^2 = 1$

$$W = \iint_R (0 - x^2)\,dA$$



$$= \int_{\theta=0}^{\theta=\pi} \int_0^1 - r^2\cos^2\theta \; r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{1} -r^2 \cos^2\theta \; r\,dr\,d\theta$$

$$= \int_0^{\pi} -\frac{1}{4}\left[r^4\right]_0^1 \cos^2\theta \, d\theta$$

$$= \int_0^{\pi} -\frac{1}{4} \cos^2\theta \, d\theta.$$

$$= \frac{-1}{2\times 4} \int_0^{\pi} 1 + \cos 2\theta \, d\theta.$$

$$= -\frac{1}{8} \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= -\frac{1}{8}\left[\pi\right] = -\frac{\pi}{8}.$$

Ex $\oint 6xy\,dx + \left[3x^2 + \ln(1+y)\right] dy \qquad C: x^2+y^2=4$

$$= \iint_R (6x - 6x)\, dA = 0$$

$$A = \iint_R 1 \cdot dA$$

$\oint_C = \iint_R \underline{1}\, dA$ gives area of enclosed region.

$$\boxed{\iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA} \rightarrow \oint P\,dx + Q\,dy$$

$$-= \frac{1}{2}\iint_R 2 \, dA$$

$$= \frac{1}{2}\iint_R 1-(-1) \, dA$$

$\frac{\partial Q}{\partial x} = 1$

$Q = x$

$$\boxed{A = \frac{1}{2}\oint_C -y\,dx + x\,dy}$$

$\frac{\partial P}{\partial y} = -1$

$P = -y$

Area of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\left.\frac{x}{a} = \cos t \quad , \frac{y}{b} = \sin t \right\} \text{parametric eqn ellipse} \qquad 0 \le t \le 2\pi$$

$x = a\cos t, \quad y = b\sin t$

$dx = -a\sin t\,dt, \quad dy = b\cos t\,dt$

$$A = \frac{1}{2}\int_0^{2\pi} -b\sin t\,(-a\sin t)\,dt + a\cos t \cdot b\cos t\,dt$$

$$= \frac{1}{2}\int ab\,(\sin^2 t + \cos^2 t)\,dt$$

$$= \frac{1}{2}\,ab\,[t]_0^{2\pi} = \underline{\pi ab}$$

Ex $\oint_C (x + e^x \sin y)\,dx + (x + e^x \cos y)\,dy$:

$C;\ \frac{x^2}{9} + \frac{y^2}{4} = 1$

$a^2 = 9 \quad a = 3$

$b^2 = 4, \quad b = 2$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

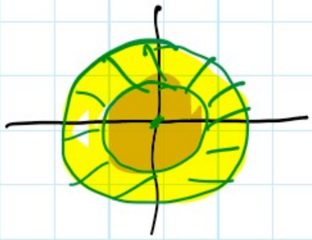
$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \qquad a=3,\ b=2$$

$$= \iint_R \left(1 + \cancel{\cos y \cdot e^x} - \cancel{e^x \cos y} \right) dA$$

$$= \iint_R 1 \, dA.$$

$$= \pi(3)(2) = \underline{\underline{6\pi}}$$

Ex $\oint_C -y \, dx + x \, dy$; C : bounded by $x^2+y^2=1$ $x^2+y^2=4$



$$= \iint_R 1 - (-1) \, dA.$$

$$= \iint 2 \, dA = 2 \iint 1 \, dA$$

$$= 2 \left[\pi(2)^2 - \pi(1)^2 \right]$$

$$W = 2 \left[4\pi - 1\pi \right] = \underline{\underline{6\pi}}$$