Line integral on conservative vector field

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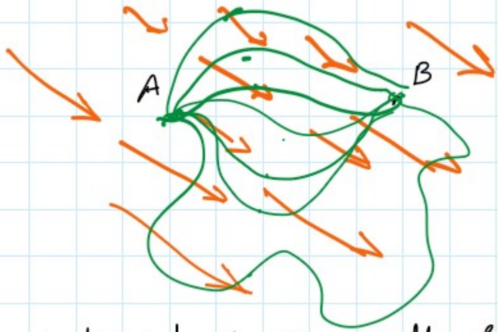
'F' is conservative iff $F = \nabla f$ for some potential 'f'.

$$W = \int_C F \cdot dr$$



The line $\int$ is independent on path so, the $\int$ will be the same, no matter what curve we choose as long as end points stay the same.

→ For conservative vector field, we don't even define a curve C.

Fundamental theorem of calculus:- II

$$\int_a^b f'(x)\,dx = f(b) - f(a) \leftarrow$$

$$F = \nabla f \longrightarrow 'f' \begin{cases} \nabla f = f_x \hat{i} + f_y \hat{j} \\ F = P\hat{i} + Q\hat{j} \end{cases}$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \quad \text{for function 'f'}$$

'f' is the antiderivative of ∇f

Let F be a conservative vector field.

$F = \nabla f$ for 'f'

$$W = \int_C F \cdot dr = \int \nabla f \, dr = \boxed{f(b) - f(a)}$$

Fundamental theorem of line integral.

1) Show 'F' is conservative,

∂Q ... ∂P ? curl=0

1) Show 'F' is conservative,

$F(x,y) = P\hat{i} + Q\hat{j}$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ } curl=0

$\text{curl} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}$

$F(x,y,z) = P\hat{i} + Q\hat{j} + R\hat{k}$

curl F = 0

✓ $\text{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k} = 0$

$\left\{ \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \ \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \ \& \ \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \right\}$

Ex work done on $F(x,y) = \underbrace{(2y+1)}_{P}\hat{i} + \underbrace{(2x+3)}_{Q}\hat{j}$ from $A(0,0)$ to $B(-1,1)$.

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$2 = 2$

The $F(x,y)$ is conservative vector field.

$\int_C F \cdot dr = \int \nabla f \, dr = f(b) - f(a)$

$F = \nabla f = (2y+1)\hat{i} + (2x+3)\hat{j}$

$\frac{\partial f}{\partial x} = 2y+1.$ $\quad \left\{ \frac{\partial f}{\partial y} = 2x+3 \right.$

$\int \frac{\partial f}{\partial x} dx = \int 2y+1 \, dx = 2xy + x + g(y)$

$\frac{\partial}{\partial y}$

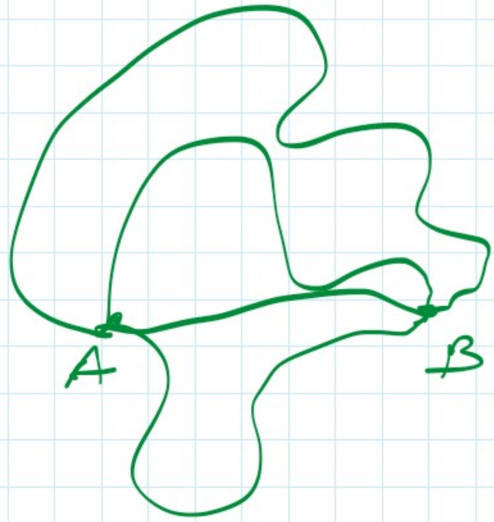
$\frac{\partial f}{\cdot} = 2x + g'(y)$

$$\frac{\partial f}{\partial y} = 2x + g'(y)$$

set equal $\longrightarrow$ $2x + 3 = 2x + g'(y)$

$$g'(y) = 3$$

Integrate $\longrightarrow$ $g(y) = 3y + C$



$$f(x,y) = \underline{2xy + x + 3y} + C$$

$$\omega = \int_C F \cdot dr = f(b) - f(a)$$

$$= 2(-1)(1) + (-1) + 3(1) + \cancel{C} - 0 - \cancel{C}$$

$$= \underline{\underline{0}}$$

Ex Find work done on $F(x,y) = xe^{2y}\,\hat{i} + x^2e^{2y}\,\hat{j}$ from $A(0,0) \to B(-1,1)$.

P Q

Solⁿ

① $\frac{\partial P}{\partial y} = 2xe^{2y}$; $\frac{\partial Q}{\partial x} = 2xe^{2y}$

$$\because \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$\therefore$ F is conservative vector field.

$F = \nabla f$ for $f(x,y)$

$$\omega = \int_C F \cdot dr = \int \nabla f \cdot dr = f(b) - f(a).$$

$$F = \nabla f = \underbrace{xe^{2y}}_{\frac{\partial f}{\partial x}}\,\hat{i} + \underbrace{x^2e^{2y}}_{\frac{\partial f}{\partial y}}\,\hat{j}$$

$$f(x,y) = \int xe^{2y}\,dx$$

$$= \frac{x^2}{2}e^{2y} + g(y)$$

$$\frac{\partial f}{\partial y} = \frac{x^2}{2} \cdot e^{2y} \cdot 2 + g'(y)$$

$$\frac{\partial f}{\partial y} = \frac{x^2}{2} \cdot e^{2y} \cdot 2 + g'(y)$$

$$= x^2 \cdot e^{2y} + g'(y)$$

$$g'(y) = 0$$

$$g(y) = C$$

$$f(x,y) = \frac{x^2}{2} e^{2y} + C$$

$$\omega = \int_C F \cdot dr = \int \nabla f \cdot dr = f(-1,1) - f(0,0)$$

$$= \frac{e^2}{2} + C - 0 - C$$

$$= \frac{e^2}{2}$$

Ex $F(x,y) = \underset{P}{\left(x^2 + \frac{y}{x}\right)}\,\hat{i} + \underset{Q}{\left(y^2 + \ln x\right)}\,\hat{j}$

$A(1,0) \to B(e,1)$

Find work done.

Soln.

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

$$\frac{1}{x} = \frac{1}{x}$$

F is conservative.

$F = \nabla f$ for $f(x,y)$.

$$\nabla f = \underset{\frac{\partial f}{\partial x}}{x^2 + \frac{y}{x}}\,\hat{i} + \underset{\boxed{\frac{\partial f}{\partial y}}}{y^2 + \ln x}\,\hat{j}$$

$$f(x,y) = \int \frac{\partial f}{\partial x}\,dx = \int x^2 + \frac{y}{x}\,dx.$$

$$= \boxed{\frac{x^3}{3} + y\ln x + g(y)}$$

$$\Big\downarrow \frac{\partial}{\partial y}$$

$$\ln x + g'(y) \xrightarrow[\text{equal}]{\text{set}} y^2 + \ln x.$$

$$g'(y) = y^2$$

$$g(y) = \int g'(y)\,dy = \frac{y^3}{3} + C.$$

$$f(x,y) = \frac{x^3}{3} + y\ln x + \frac{y^3}{3} + c.$$

$$\omega = f(e,1) - f(1,0) = \frac{e^3}{3} + 1$$

Ex $$F(x,y) = \underbrace{(x + \tan^{-1}y)}_{P}\,\hat{i} + \underbrace{\left(\frac{x+y}{1+y^2}\right)}_{Q}\hat{j}$$

$$\frac{\partial Q}{\partial x} = \frac{1}{1+y^2}$$

$$\frac{\partial P}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial y}$$; f is conservative

$$F = \nabla f.$$

$$f(x,y) = \int \frac{\partial f}{\partial x}\,dx = \int x + \tan^{-1}y\,dx.$$

$$= \underbrace{\frac{1}{2}x^2 + x\tan^{-1}y}_{\text{Derivative wrt } y} + g(y)$$

$$\frac{\partial f}{\partial y} = \frac{x}{1+y^2} + g'(y)$$

$$\frac{x}{1+y^2} + g'(y) = \frac{x+y}{1+y^2}$$

$$g'(y) = \frac{y}{1+y^2}$$

$\xrightarrow{\text{Integrate}}$ $$g(y) = \int g'(y)\,dy = \int \frac{y}{1+y^2}\,dy$$

$$= \frac{1}{2}\ln(1+y^2) + c.$$

$$f(x,y) = \frac{1}{2}x^2 + x\tan^{-1}y + \frac{1}{2}\ln(1+y^2) + c.$$

Ex $$F(x,y,z) = \underbrace{\cos y}_{P}\,\hat{i} + \underbrace{(z^2 - x\sin y)}_{Q}\,\hat{j} + \underbrace{2yz}_{R}\,\hat{k}$$

Ex $F(x,y,z) = \cos y\ \hat{i} + (z^2 - x\sin y)\hat{j} + 2yz\ \hat{k}$

from $(1,0,0) \to (2, 2\pi, 1)$

Sol^n

$$\text{Curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= (2z - 2z)\hat{i} - (0-0)\hat{j} + (-\sin y + \sin y)\hat{k}$$

$$= 0.$$

F is Conservative $\quad F = \nabla f.$

$$f(x,y,z) = \int \frac{\partial f}{\partial x} dx = \int \cos y\, dx$$

$$= x\cos y + g(y,z)$$

Derivate $\rightarrow \quad \frac{\partial f}{\partial y} = -x\sin y + g'(y,z)$

$$= z^2 - x\sin y.$$

$$g'(y,z) = z^2$$

$$g(y,z) = \int z^2\, dy = yz^2 + h(z)$$

$$f(x,y,z) = x\cos y + yz^2 + h(z)$$

Derivate wrt z $\rightarrow \quad \frac{\partial f}{\partial z} = 2yz + h'(z) = 2yz \left(\text{given } \frac{\partial f}{\partial z}\right)$

$$h'(z) = 0.$$

$$h(z) = C.$$

$$f(x,y,z) = x\cos y + yz^2 + C.$$

$$w = f(2, 2\pi, 1) - f(1,0,0)$$

$$= 2\cos 2\pi + 2\pi(1)^2 + C - 1\cos 0 - 0(0)^2 - C$$

$$= 2 + 2\pi - 1$$

$$w = 1 + 2\pi$$

Ex work done on $F(x,y,z) = yz^2\ \hat{i} + xz^2\ \hat{j} +$

Ex work done on $F(x,y,z) = yz^2\,\hat{i} + xz^2\,\hat{j} + 2xyz\,\hat{k}$

from $(0,0,1) \rightarrow (1,3,2)$.

F is conservative

$$F = \nabla f$$

$$f(x,y,z) = \int \frac{\partial f}{\partial x}\,dx = \int y z^2\,dx$$

$$= \boxed{xyz^2} + g(y,z)$$

$$\frac{\partial f}{\partial y} = xz^2 + g'(y,z) \quad , \quad g'(y,z) = 0$$

$$g(y,z) = C$$

$$f(x,y,z) = xyz^2 + C$$

$$\frac{\partial f}{\partial z} = \underline{2xyz}$$

$$f(x,y,z) = xyz^2$$

$$W = f(1,3,2) - f(0,0,1) = 12$$