Change of variable in Multiple integral

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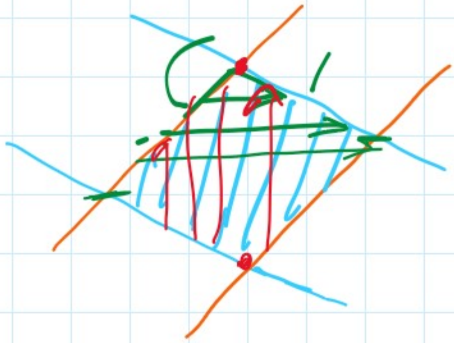
opening problem:-

$\iint_R (x+y)\, dA$, 'R', region bound by

$y=-2x$, $y=\frac{1}{2}x-\frac{15}{2}$, $y=-2x+10$,

$y=\frac{1}{2}x$.



For $\int_{x=a}^{b} f(x)\, dx$. $[a,b] \in$ x-axis.

$x = g(u)$

$\int_c^d f(g(u))\, g'(u)\, du$.

$[c,d] \longrightarrow$ u-axis

For $\iint_R$, we integrate over a region.

$R \in (x,y)$

$x=g(u,v)$, $y=h(u,v)$

$S \in (u,v)$

$$\iint_R f(x,y)\,dA \to \iint_S f(g(u,v),h(u,v))\,J \cdot du\,dv$$

'J' Jacobian Matrix $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

In polar :

$x = r\cos\theta$. $\qquad x^2+y^2=r^2$.

$y = r\sin\theta$,

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - [-r\sin^2\theta]$$

$$= r[\cos^2\theta + \sin^2\theta] = \underline{r}$$

Ex $\underbrace{x = 2u+v \quad \& \quad y = u^2-v}$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 2u & -1 \end{vmatrix} = -2-2u$$

$$= \underline{\underline{-2(1+u)}}$$

J

Ex: $$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$x = 2u + w, \quad y = u^2 - v^2, \quad z = u + v^2 - 2w^2$

$$J = \begin{vmatrix} 2 & 0 & 1 \\ 2u & -2v & 0 \\ 1 & 2v & -4w \end{vmatrix} = 2(8vw - 0) - 0(\ \ ) + 1(4uv + 2v)$$

$$= 16vw + 4uv + 2v$$

Ex: $\iint_R (x+y)\, dA$ 'R' $\left\{ \begin{array}{l} y = -2x, \ y = \frac{1}{2}x - \frac{15}{2}, \\ y = -2x + 10, \ y = \frac{1}{2}x. \end{array} \right\}$

Transformation $x = u + 2v$, $y = v - 2u$

Soln.

a) $y = -2x \Rightarrow v - 2u = -2(u + 2v)$

$v - \cancel{2u} = -\cancel{2u} - 4v$

$v + 4v = 0$

$5v = 0$

$\Rightarrow \underline{v = 0}$

b) $y = \frac{1}{2}x - \frac{15}{2} \Rightarrow 2y = x - 15$

$2(v - 2u) = u + 2v - 15$

$\Rightarrow \cancel{2v} - 4u = u + \cancel{2v} - 15$

$\Rightarrow -5u = -15 \Rightarrow \underline{u = 3}$

c) $y = -2x + 10 \Rightarrow v - 2u = -2(u + 2v) + 10$

$\Rightarrow v - \cancel{2u} = -\cancel{2u} - 4v + 10$

$\Rightarrow 5v = 10 \Rightarrow \underline{v = 2}$

d) $y = \frac{1}{2}x \Rightarrow 2y = x \Rightarrow 2(v-2u) = u+2v$

$\Rightarrow \cancel{2v} - 4u = u + \cancel{2v}$

$\Rightarrow 5u = 0$

$\Rightarrow \underline{\underline{u = 0}}$

$$\iint_R (x+y)\, dA = \int_{v=0}^{2} \int_{u=0}^{u=3} [\underbrace{u+2v}_{x} + \underbrace{v-2u}_{y}]\, 5\, du\, dv$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1 - (-4) = \underline{5}$$

$$= 5 \int_{v=0}^{2} \int_{u=0}^{u=3} [u + 2v + v - 2u]\, du\, dv$$

$$= 5 \int_{v=0}^{2} \int_{u=0}^{3} (3v - u)\, du\, dv$$

$$= 5 \int_{v=0}^{2} \left[3uv - \frac{1}{2}u^2\right]_0^3 dv$$

$$= 5 \int_{v=0}^{2} 9v - \frac{9}{2}\, dv = 5\left[\frac{9}{2}v^2 - \frac{9}{2}v\right]_0^2$$

$$= 5(18-9) = 45$$

Ex: $\iint \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}\, dA$, 'R'

Ex: $\iint_R \sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}\; dA$, R

$\frac{x^2}{4}+\frac{y^2}{9}=1.$

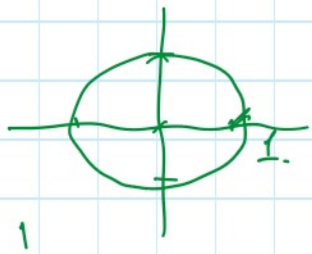
ellipse..

Transformation: $x = 2u$, $y = 3v$

Soln:

$$\frac{(2u)^2}{4}+\frac{(3v)^2}{9}=1$$

$$u^2+v^2=1.$$ — circle.



$$\iint_R \underbrace{\sqrt{1-\left(\frac{x^2}{4}+\frac{y^2}{9}\right)}}\, dA = \underbrace{\int_{\theta=0}^{2\pi}\int_{r=0}^{1}}_{\text{intermediate steps.}} \sqrt{1-r^2}\; J\, r\, dr\, d\theta$$

$x=2u$, $y=3v$

$$J=\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}$$

$$=6$$

$$= \int_0^{2\pi}\int_{r=0}^{1} \sqrt{1-r^2}\; 6\cdot r\, dr\, d\theta.$$

$$= 6\int_0^{2\pi}\int_{r=0}^{1} \sqrt{1-r^2}\; \underline{r}\, dr\, d\theta.$$

$r=0, w=1$

$\int_0 \int_{r=0}$

$1 - r^2 = w$

$-2r\,dr = dw$

$r=0, w=1$

$r=1, w=0$

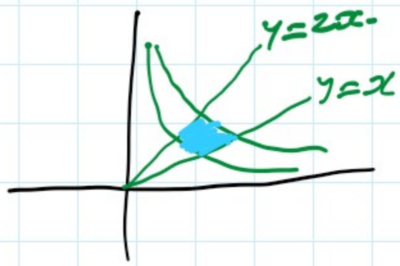
$$= \frac{6}{2} \int_0^{2\pi} \int_{w=0}^{1} \sqrt{w}\, dw\, d\theta.$$

$$= 3 \int_0^{2\pi} \left[\frac{2w^{3/2}}{3}\right]_0^1 d\theta$$

$$= 3 \int_0^{2\pi} \frac{2}{3}\, d\theta = 3\left[\frac{2}{3}\theta\right]_0^{2\pi}$$

$$= 3 \times \frac{4\pi}{3} = \underline{\underline{4\pi}}$$

Ex: $\iint_R xy^2\, dA$, 'R' $xy=1$, $xy=2$, $y=x$, $y=2x$.



Transformation

$x = \frac{u}{v}$, $y = v$

1) $xy = 1. \Rightarrow \frac{u}{v}\cdot v = 1 \Rightarrow u = 1.$

2) $xy = 2 \Rightarrow \frac{u}{v}\cdot v = 2 \Rightarrow u = 2$

3) $y = x \Rightarrow v = \frac{u}{v} \Rightarrow v^2 = u \Rightarrow v = \sqrt{u}$

4) $y = 2x \Rightarrow v = \frac{2u}{v} \Rightarrow v^2 = 2u \Rightarrow v = \sqrt{2u}$

$$\int_{u=1}^{u=2}\int_{v=\sqrt{u}}^{v=\sqrt{2u}} \frac{u}{v}\cdot v^2 \; J \, dv\, du$$

$x=\frac{u}{v},\ y=v$

$$J = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

$$\int_{u=1}^{u=2}\int_{v=\sqrt{u}}^{v=\sqrt{2u}} u \; dv\, du$$

$$= \int_{u=1}^{u=2} u\,[v]_{\sqrt{u}}^{\sqrt{2u}}\, du = \int_1^2 u\cdot(\sqrt{2u}-\sqrt{u})\,du$$

$$= \int_1^2 \sqrt{2}\, u^{3/2} - u^{3/2}\, du$$

$$= \left[\sqrt{2}\times 2\frac{u^{5/2}}{5} - \frac{2u^{5/2}}{5}\right]_1^2$$

$$= \left(\frac{2\sqrt{2}}{5} - 1\right)\left[2^{5/2} - 1^{5/2}\right]$$

$$= \left(\frac{2\sqrt{2}}{5}-1\right)\left(2^{5/2}-1\right)$$

$$= \frac{2}{5}\left(9-5\sqrt{2}\right)$$

Ex. 'R' $x+y=-1$, $x+y=3$, $2x-y=0$, $2x-y=4$

Transformation: $x+y=u$ & $2x-y=v$

$u=-1,\ u=3,\ v=0,\ v=4$

Rectangular to Spherical co-ordinate system.

$$J = \rho^2 \sin\phi.$$

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \sin\phi\cos\theta\left(\rho^2\sin^2\phi\cos\theta\right) - \rho\cos\phi\cos\theta\left(-\rho\sin\phi\cos\phi\cos\theta\right)$$
$$- \rho\sin\phi\sin\theta\left(-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta\right)$$

$$= \rho^2\sin^3\phi\cos^2\theta + \rho^2\sin\phi\cos^2\phi\cos^2\theta +$$
$$\rho^2\sin^3\phi\sin^2\theta + \rho^2\sin\phi\cos^2\phi\sin^2\theta.$$

$$= \rho^2\sin^3\phi \times 1. + \rho^2\sin\phi\cos^2\phi \times 1.$$

$$= \rho^2\sin\phi\left[\sin^2\phi + \cos^2\phi\right]$$

$= \underline{\underline{\rho^2 \sin\phi}}$ $\underline{\underline{\text{Jacobian}}}$