Spherical coordinate system

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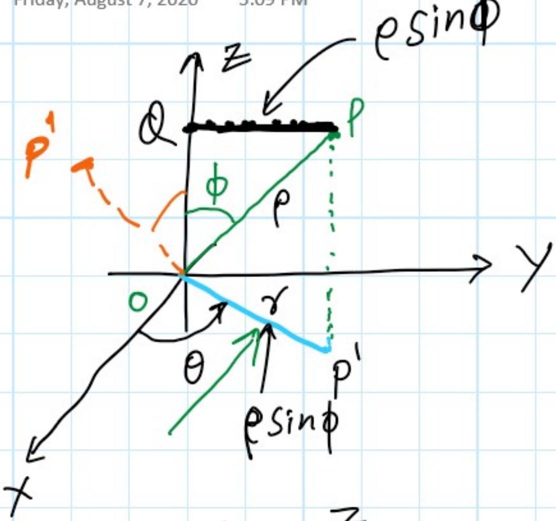
$$\{ x^2+y^2+z^2=\rho^2 \}$$



$$r = \rho \sin\phi$$

$$\left\{\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta. \\ z &= \rho \cos\phi \end{aligned}\right\}$$

$$\cos\phi = \frac{z}{\rho}$$

$$z = \rho \cos\phi$$

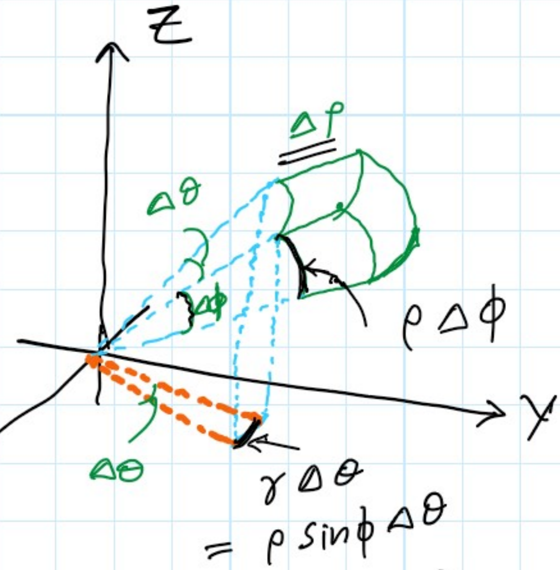
$$\left\{\begin{aligned} 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{aligned}\right\}$$

$$x^2+y^2+z^2=\rho^2$$

$$\iiint_T f(x,y,z)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{\phi=0}^{\phi}\int_{\rho=}^{\rho=} f(x,y,z)\,\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

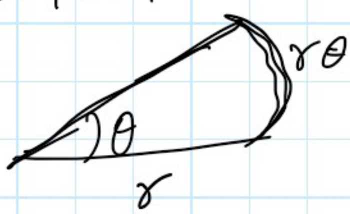
Spherical wedge:-

V = Arc length along θ × Arc length along φ × depth.



$$\Delta V = \rho \sin\phi\, \Delta\theta \times \rho\, \Delta\phi \times \Delta\rho$$

$$\Delta V = \rho^2 \sin\phi \; \Delta\rho\, \Delta\phi\, \Delta\theta.$$



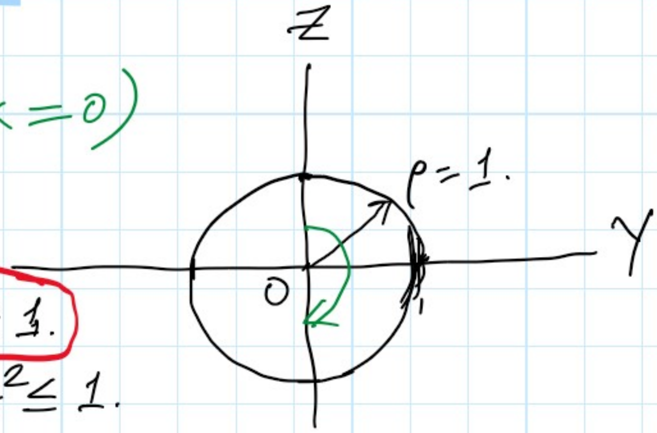
$$\Delta \to 0$$

$$\iiint_D dV = \iiint_D \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta.$$

Ex: $\iiint_T \sqrt{x^2+y^2+z^2}\, dv$, 'T' the sphere $x^2+y^2+z^2 \le 1$.

Hw $\quad x^2+y^2+z^2 \le 1.$

R-3 Simple: $\quad \rho$ - simple. $(x=0)$



$\rho^2 \le 1$

$0 \le \rho \le 1.$

$\rightarrow x^2+y^2+z^2 \le 1.$

$\xrightarrow{x=0} y^2+z^2 \le 1.$

$0 \le \rho \le 1$

ϕ - simple

$0 \le \phi \le \pi$

θ - simple: $(z=0)$

$x^2+y^2 \le 1.$

$0 \le \theta \le 2\pi$

$x^2+y^2+z^2 = \rho^2$

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{\rho=1} \rho \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \left[\frac{\rho^4}{4}\right]_0^1 \sin\phi \, d\phi \, d\theta.$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{1}{4} \sin\phi \, d\phi \, d\theta.$$

$$= \int_0 \int_0 \quad 4$$

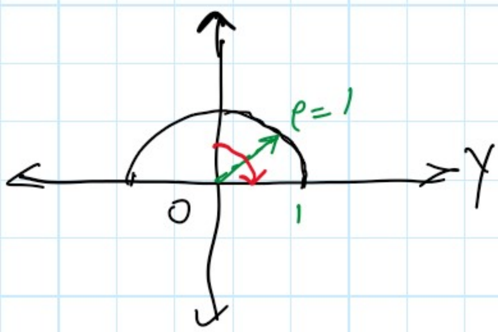
$$= \int_0^{2\pi} -\frac{1}{4} \left[\cos\phi\right]_0^{\pi} d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \, d\theta = \frac{1}{2}\left[\theta\right]_0^{2\pi} = \pi$$

Ex: $\iiint_T y \, dv$, 'T' region bound by $z = \sqrt{1-x^2-y^2}$ & xy-plane.

Sol^n: $z = \sqrt{1-x^2-y^2}$

$$x^2+y^2+z^2 = 1.$$



ρ-simple: (set $x=0$)

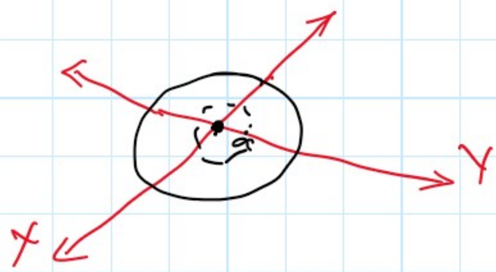
$$z = \sqrt{1-y^2}$$

$$0 \le \rho \le 1$$

φ-simple: $0 \le \phi \le \pi/2$

θ-simple: $(z=0)$

$$0 \le \theta \le 2\pi$$



$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{1} \rho \sin\phi \sin\theta \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int^{2\pi} \int^{\pi/2} \frac{1}{4} \sin^2\phi \sin\theta \left[\rho^4\right]_0^1 d\phi \, d\theta$$

$$= \int_{0}^{2\pi} \int_{\phi=0}^{\pi/2} \frac{1}{4} \sin^2\phi \sin\theta \left[\;\right]_0 \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{4} \sin^2\phi \sin\theta \, d\phi \, d\theta$$

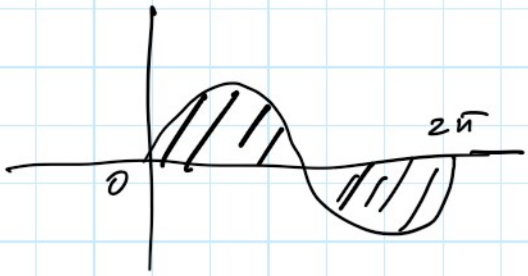
$$= \int_0^{2\pi} \frac{1}{8} \left[\phi - \frac{\sin 2\phi}{2}\right]_0^{\pi/2} \sin\theta \, d\theta .$$

$$= \boxed{\int_0^{2\pi}} \frac{1}{8} \left[\frac{\pi}{2} - 0\right] \sin\theta \, d\theta$$

$$= \frac{\pi}{16} \left[-\cos\theta\right]_0^{2\pi}$$

$$= \frac{\pi}{16} \left[-\cos 2\pi + \cos 0\right] = \underline{\underline{0}}$$

$$\int_0^{2\pi} \rightarrow 2 \int_0^{\pi} = 0 \longrightarrow 4 \int_0^{\pi/2} = 0$$



Ex: $\iiint_T \underline{\underline{xz}} \, dV$, 'T' Solid bound by

$x^2+y^2+z^2=4$ & $z=\sqrt{x^2+y^2}$

Soln, $x^2+y^2+z^2=4$, $z=\sqrt{x^2+y^2}$

Sol^n:

$x^2+y^2+z^2=4$, $z=\sqrt{x^2+y^2}$

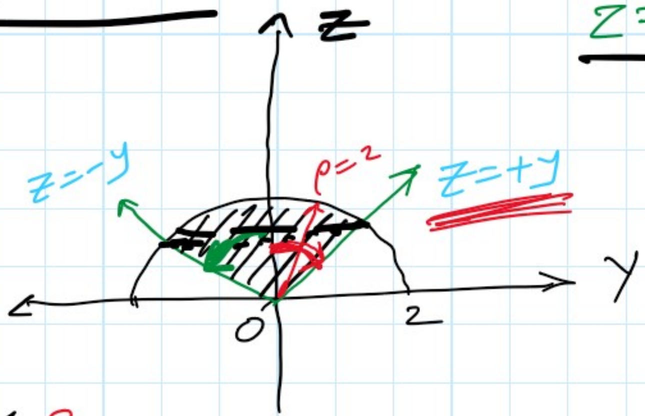
Sphere   cone.

ρ-simple: $(x=0)$

$y^2+z^2=4$

$z=\sqrt{y^2}$

$z=\pm y$



$\tan^{-1}\left(\frac{z}{y}\right)$

$\tan^{-1}\left(\frac{y}{y}\right)=\pi/4$

$0\le\rho\le 2$

ϕ-simple: $0\le\phi\le\frac{\pi}{4}$

θ-simple: $(z=0)$

$0\le\theta\le 2\pi$

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{\rho=2}(\underbrace{\rho\sin\phi\cos\theta}_{x}\cdot\underbrace{\rho\cos\phi}_{z})\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$=\underline{0}$

Ex   volume of solid bound by

$z=2$ & $z=\sqrt{x^2+y^2}$

Sol^n

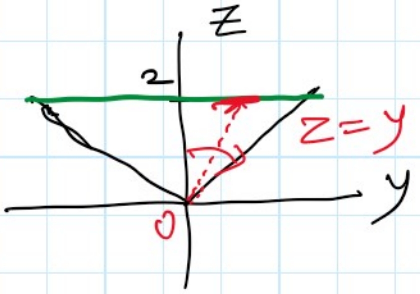
$z=2$ & $z=\sqrt{x^2+y^2}$

Soln:

$$z=\sqrt{x^2+y^2}$$

ρ-simple: $(x=0)$

$$z=\sqrt{y^2} \Rightarrow z=\pm y.$$



$$\cancel{z}=2$$
$$\rho\cos\phi=2$$
$$\rho=\frac{2}{\cos\phi}=2\sec\phi$$

$$0\le\rho\le 2\sec\phi$$

ϕ-simple: $\quad 0\le\phi\le \pi/4$

θ simple: $\quad 0\le\theta\le 2\pi$

$$V=\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{2\sec\phi}(1)\ \rho^2\sin\phi\ d\phi\, d\theta.$$

$$=\pi/3$$

Ex: Find mass of above solid where mass density proportional to the square of the distance from origin.

Soln.

$$\sigma(x,y,z)=k\cdot\rho^2$$

$$\int^{2\pi}\int^{\pi/4}\int^{2\sec\phi} k\cdot\rho^2\cdot\ \rho^2\sin\phi\ d\rho\, d\phi\, d\theta.$$

$$m = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{2\sec\phi} k \cdot \rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{48\pi k}{5} \leftarrow \underline{\underline{\text{Ans}}}$$

<u>center of mass</u>:. lies on z axis.

$$(\bar{x}, \bar{y}, \bar{z}) \equiv (0, 0, \bar{z})$$

$$\bar{z} = \frac{M_{xy}}{M}$$

$$M_{xy} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\sec\phi} \underbrace{\rho\cos\phi}_{z} \; k\rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 16\pi k.$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{16\pi k}{\frac{48\pi k}{5}} = \frac{5}{3}.$$

<u>Ex:</u> Use spherical coordinate to find the volume of solid that lies above the cone $z = \sqrt{x^2+y^2}$ & below the sphere $x^2+y^2+z^2 = z$.

<u>Soln.</u> ρ-simple $(x=0)$

$$y^2+z^2 = z$$

Third term

$$\Rightarrow y^2 + z^2 - z = 0$$

$$\Rightarrow y^2 + z^2 - z + \frac{1}{4} = \frac{1}{4}$$

$$\Rightarrow y^2 + \left(z - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

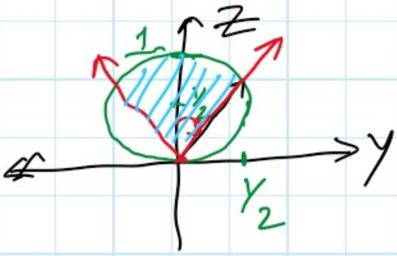
circle $r = 1/2$ & center $(0, \frac{1}{2})$

Circle $r = 1/2$ & center $(0, 1/2)$

$z = \sqrt{x^2+y^2}$

$\downarrow x = 0$

$z = \sqrt{y^2}$

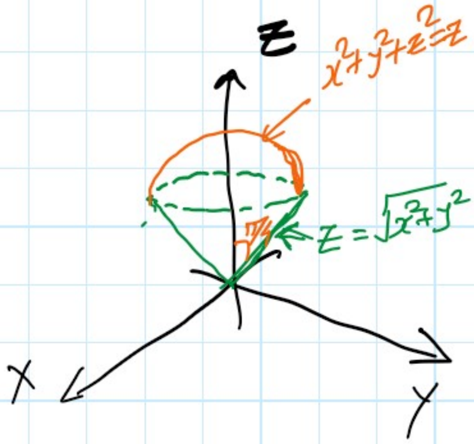


$x^2+y^2+z^2 = z$

$\rho^2 = \rho\cos\phi$

$\Rightarrow \rho = \cos\phi$

$0 \le \rho \le \cos\phi$



$0 \le \phi \le \pi/4$

θ-simple: $0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} (1) \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3}\right]_0^{\cos\phi} d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \sin\phi \cos^3\phi \, d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \sin\phi \, (1-\sin^2\phi) \cos\phi \, d\phi$$

$u = \sin\phi$
$du = \cos\phi \, d\phi$

$$= \frac{2\pi}{3} \int_0^{1/\sqrt{2}} u(1-u^2) \, du$$

$$\frac{2\pi}{} \left[\frac{u^2}{} - \frac{u^4}{} \right]^{1/\sqrt{2}}$$

$$= \frac{2\pi}{3}\left[\frac{u^2}{2} - \frac{u^4}{4}\right]_0^{1/\sqrt{2}}$$

$$= \frac{2\pi}{3}\left[\frac{1}{4} - \frac{1}{16} - 0\right]$$

$$= \frac{2\pi}{3} \times \frac{3}{16} = \frac{\pi}{8}.$$