Triple Integral for Cylindrical Co-ordinate System

Friday, July 31, 2020 10:56 AM

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

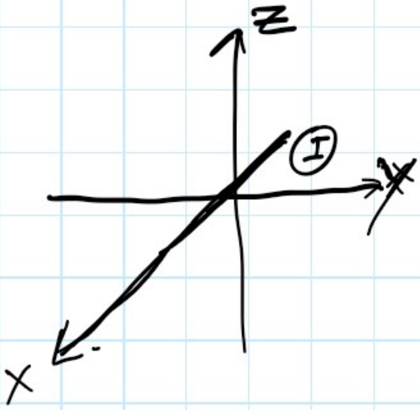
& $x^2+y^2=r^2$

$f(x,y,z) = f(r\cos\theta, r\sin\theta, z)$

$$\iiint_T f(x,y,z)\, dV = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\int_{z_1}^{z_2} f(r\cos\theta, r\sin\theta, z)\, dz \cdot r \cdot dr\, d\theta$$

R-3- z-simple.
'R' xy-plane.

Ex $\iiint_T y\, dV$, 'T' solid bounded by $z = 4-x^2-y^2$ in first octant.



Soln:

$0 \le z \le 4-x^2-y^2$

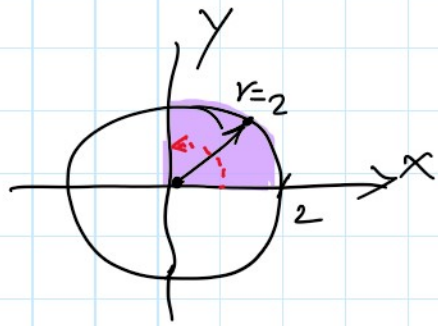
$0 \le z \le 4-r^2$

Intersection:

$z=0$, & $z=4-x^2-y^2$

$0 = 4-x^2-y^2$

$x^2+y^2=4$



$0 \le r \le 2$

$0 \le \theta \le \pi/2$

$y = r\sin\theta$

$$= \int_0^{\theta=\pi/2}\int_0^{r=2}\int_0^{z=4-r^2} y\; r\, dz\, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2} \int_{z=0}^{z=4-r} y \; r \, dz \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} \int_{z=0}^{4-r^2} r^2 \sin\theta \, dz \, dr \, d\theta$$

ANS $= \frac{64}{15}$

$=$

Ex: find volume of 'T' solid bound by

$x^2+y^2+z^2=9$ & $8z = x^2+y^2$

Soln: R-3 ; z-simple.

$z = \sqrt{9-x^2-y^2}$ & $z = \frac{1}{8}(x^2+y^2)$

$(0,0) \rightarrow z = \sqrt{9} = 3$ $\quad z=0$

$\frac{1}{8}(x^2+y^2) \le z \le \sqrt{9-x^2-y^2} \Rightarrow \frac{1}{8} r^2 \le z \le \sqrt{9-r^2}$

'R' on XY-plane:

$x^2+y^2+z^2=9$ & $8z = x^2+y^2$

$8z + z^2 = 9$

$z^2+8z-9=0$

$(z+9)(z-1)=0.$

$z=-9$ & $z=1.$ $\quad z=1$

$\times$

$\boxed{z=1}$

$8(1) = x^2+y^2$

$x^2+y^2=8 \longrightarrow$

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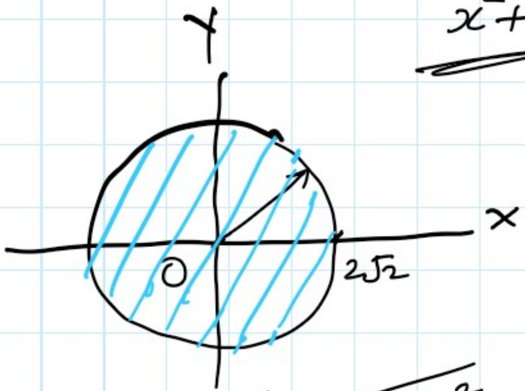


$0 \le r \le 2\sqrt{2}$

$0 \le \theta \le 2\pi$

volume

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{r=2\sqrt{2}} \int_{z=\frac{1}{8}r^2}^{z=\sqrt{9-r^2}} 1 \; r \, dz \, dr \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{2\sqrt{2}} r \, \left[z \right]_{z=\frac{1}{8}r^2}^{z=\sqrt{9-r^2}} \, dr \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{2\sqrt{2}} \left[r\sqrt{9-r^2} - r \cdot \frac{1}{8} r^2 \right] dr \, d\theta$$

u-sub.

$$= \int_{0}^{2\pi} \int_{0}^{2\sqrt{2}} r \sqrt{9-r^2} \, dr \, d\theta - \int_{0}^{2\pi} \int_{0}^{2\sqrt{2}} \frac{1}{8} r^3 \, dr \, d\theta$$

$9-r^2 = u$ | $r=0,\ u=9$

$-2r\,dr = du$ | $r=2\sqrt{2},\ u=1$

$r\,dr = -\frac{1}{2} du$

$$= \int_{0}^{2\pi} \int_{1}^{9} \frac{1}{2} u^{1/2} \, du \, d\theta - \int_{0}^{2\pi} \left[\frac{1}{32} r^4 \right]_{0}^{2\sqrt{2}} d\theta$$

$$= \int_{0}^{2\pi} \frac{1 \times 2}{2 \times 3} \left[u^{3/2} \right]_{1}^{9} d\theta - \int_{0}^{2\pi} 2 \, d\theta$$

$2\pi\left(\frac{16\sqrt{2}}{3} - 2\right)$

$$= \int_0^{2\pi} \frac{27}{3} - \frac{1}{3}\, d\theta - \int_0^{2\pi} 2\, d\theta \qquad \left(9 - \frac{1}{3} - 2\right)$$

$$= \int_0^{2\pi} \frac{20}{3}\, d\theta = \frac{20}{3}\left[\theta\right]_0^{2\pi} \qquad \left(7 - \frac{1}{3}\right)$$

$$= \frac{40\pi}{3}. \qquad \frac{20}{3}$$

COM :- $(\bar{x}, \bar{y}, \bar{z})$

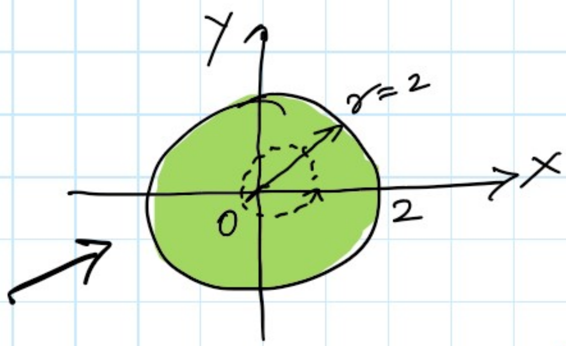
$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

Ex Find COM of solid bound by $x^2+y^2=4$, $z=0$, $z=3$, where the mass density at a point is directly proportional to the points distance from xy-plane.

Soln:- Mass density f^n. $\rho(x,y,z) = k \cdot z$

R^3 (T) z simple $\quad 0 \le z \le 3$

'R' $\quad x^2+y^2=4 \quad (z=0)$



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

kz

$$M = \int^{2\pi}\int^{2}\int^{3} \rho(x,y,z)\, r\, dz\, dr\, d\theta \quad ✓$$

$$M = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{3} \rho(x,y,z)\, r \; dz\, dr\, d\theta$$

$$= 18K\pi$$

$\bar{x}=0, \bar{y}=0, \bar{z}=?$

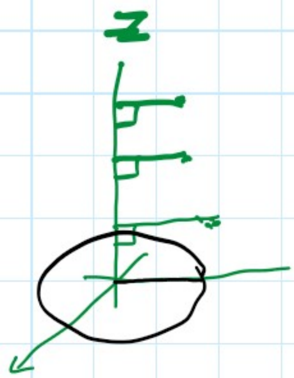
$M_{xy} = 36K\pi$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{36K\pi}{18K\pi} = 2.$$

Ex: Find COM of solid bound by $x^2+y^2=4$, $z=0$, $z=3$ where the mass density at a point is directly proportional to the points distance from z axis. $\rho(x,y,z) = k \cdot r$*

Ex:

$$M = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{3} k \cdot r \; r\, dz\, dr\, d\theta$$



$\bar{x}=0, \bar{y}=0$

$\bar{z} = ?$

$M_{xy} = ?$

$$M_{xy} = \iiint_D z \cdot kr \cdot r\, dz\, dr\, d\theta$$

Ex: Mass density is uniform throughout the solid (A homogeneous solid).

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Mass density is constant

$$\rho(x,y,z) = K.$$

Ex: Find COM, for the solid bound by $z = 4 - x^2 - y^2$ & $z = 0$ that has a homogenous density.

Soln.

$$\rho(x,y,z) = K.$$

$$M = \int_{\theta=0}^{2\pi} \int_{r=0}^{r=2} \int_{z=0}^{4-r^2} K \cdot r \, dz \, dr \, d\theta.$$

$$M = \int_{0}^{2\pi} \int_{r=0}^{2} Kr \, [z]_0^{4-r^2} \, dr \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{2} Kr(4-r^2) \, dr \, d\theta$$

$$= \int_{0}^{2\pi} \left[2Kr^2 - \frac{1}{4}Kr^4\right]_0^2 d\theta$$

$$= \int_{0}^{2\pi} 4K \, d\theta = 4K[\theta]_0^{2\pi} = 8\pi K.$$

'R' is symmetry about x & y axis. $\bar{x} = 0, \bar{y} = 0$ & $\bar{z} = ?$

$$M_{xy} = \int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{4-r^2} z \cdot Kr \, dz \, dr \, d\theta.$$

$$= \frac{32K\pi}{3}$$

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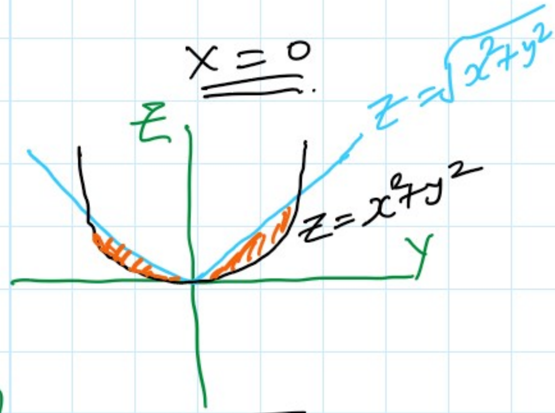
$$\bar{z} = \frac{M_{xy}}{m} = \frac{32\pi K}{3 \times 8\pi K} = \frac{4}{3}.$$

$$COM \equiv \left(0, 0, \frac{4}{3}\right).$$

Ex Find second moment of inertia about z axis of a homogenous solid bound by

$$z = \sqrt{x^2+y^2} \quad \& \quad z = x^2+y^2$$

Soln: R-3 - z-simple:



$$x^2+y^2 \le z \le \sqrt{x^2+y^2}$$

$$r^2 \le z \le r$$

'R' Draw 'R' on xy plane:

$$z = x^2+y^2, \quad z = \sqrt{x^2+y^2}$$

$$z = r^2, \quad z = \sqrt{r^2} = r$$

① $\xrightarrow{x=0}$ $z = \sqrt{x^2+y^2}$

$$z = \sqrt{y^2}$$

$$z = \pm y$$

② $z = x^2+y^2$

$\xrightarrow{x=0}$ $z = y^2$

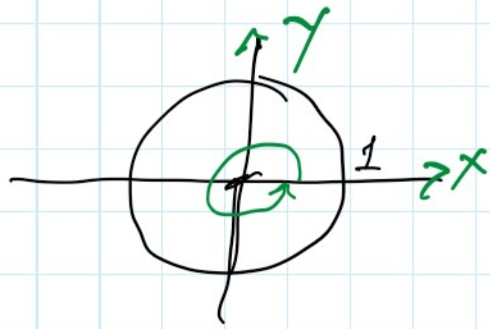
$$r^2 = r$$

$$r^2 - r = 0$$

$$r(r-1) = 0$$

$$r = 0, \quad r = 1$$

$$\left.\begin{array}{l} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{array}\right\}$$



$$x^2+y^2 = \sqrt{x^2+y^2} \leftarrow$$

$x^2+y^2 = \sqrt{x^2+y^2}$

$$(x^2+y^2)^2 = (x^2+y^2)$$

$$(x^2+y^2)^2 - (x^2+y^2) = 0.$$

$$(x^2+y^2)(x^2+y^2-1) = 0.$$

$$x^2+y^2 = 0. \quad \& \quad x^2+y^2 = 1.$$

$$I_z = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r^2}^{r} (x^2+y^2)\, k \cdot r \, dz \, dr \, d\theta$$

$$I_z = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r^2}^{z=r} k r^3 \, dz \, dr \, d\theta$$

$$= \frac{k\pi}{15}$$