Application of Triple Integration

Wednesday, July 29, 2020 10:57 AM

Ex $\rho(x,y,z) = z$, 'T' $x^2+z^2=4$, $x=2y$, $y=0$, $z=0$, $x \geq 0$

Sol^n:- R3- Try z - simple

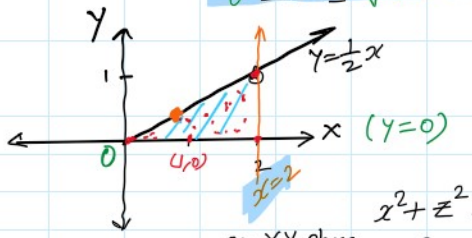
$x^2+z^2=4$

$\Rightarrow z = \sqrt{4-x^2}$

$z=0$

$0 \leq z \leq \sqrt{4-x^2}$

'R'



$x=2y$

$y=\frac{1}{2}x$

$x^2+z^2=4$

$\xrightarrow[z=0]{\text{on XY-plane}}$ $x^2=4$

$x=\pm 2$

$\because x \geq 0$, $x=2$

$0 \leq y \leq \frac{1}{2}x$

$0 \leq x \leq 2$

-H.W

$$\boxed{M = \int_{x=0}^{2}\int_{y=0}^{\frac{1}{2}x}\int_{z=0}^{\sqrt{4-x^2}} z \; dz\, dy\, dx.} = 1.$$

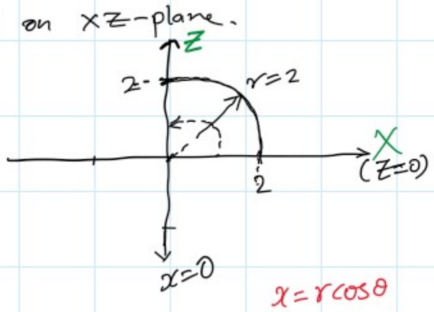
Alternative Sol^n: R3- y-simple.

$x=2y$

$y=\frac{1}{2}x$

$0 \leq y \leq \frac{1}{2}x.$

'R' is on XZ-plane.



$x^2+z^2=4$

$x=2y$

$\xrightarrow[y=0]{\text{on XZ plane}}$

$x=0.$

$x \geq 0$

$z=0$

$x=r\cos\theta$

$z=r\sin\theta$

$0 \leq r \leq 2$

$0 \leq \theta \leq \pi/2$

$$M = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2}\int_{y=0}^{\frac{1}{2}x} z \quad dy\, r.\, dr\, d\theta.$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{2}\int_{y=0}^{\frac{1}{2}r\cos\theta} r\sin\theta \cdot r \quad dy\, dr\, d\theta.$$

$$= \int_0^{\pi/2}\int_0^2 r^2 \sin\theta \, [y]_0^{\frac{1}{2} r\cos\theta} \, dr\, d\theta.$$

$$= \int_0^{\pi/2}\int_0^2 \frac{r^2}{2} \cdot \sin\theta \;\; r\cos\theta \, dr\, d\theta.$$

$$= \int_0^{\pi/2}\int_0^2 \frac{1}{2} r^3 \sin\theta\cos\theta \, dr\, d\theta.$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{8}\right]_0^2 \sin\theta\cos\theta \, d\theta$$

$$= \int_0^{\pi/2} 2\sin\theta\cos\theta \, d\theta. = \int_0^{\pi/2} \sin 2\theta \, d\theta.$$

$$= \frac{-1}{2}\left[\cos 2\theta\right]_0^{\pi/2} = -\frac{1}{2}\left[\cos\pi - \cos 0\right]$$

$$= -\frac{1}{2}\left[-1-1\right] = \underline{\underline{1}}$$

Ex $\rho(x,y,z) = \sqrt{x^2+z^2}$, 'T' Region bound by

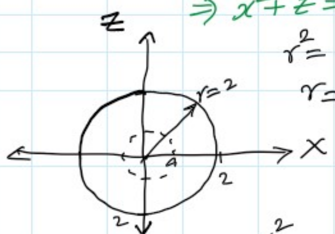
$y_1 = x^2+z^2$ & $y_2 = 8-x^2-z^2$

Solⁿ: ① R-3 :- y-simple!

$$x^2+z^2 \le y \le 8-x^2-z^2$$
$$r^2 \le y \le 8-r^2$$

② 'R' — xz plane Intersection:

$$x^2+z^2 = 8-x^2-z^2$$
$$\Rightarrow 2x^2+2z^2 = 8$$
$$\Rightarrow x^2+z^2 = 4$$
$$r^2 = 4$$
$$r = 2$$



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$\rho = \sqrt{x^2+y^2}$
$= r$

$$M = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}\int_{y=r^2}^{y=8-r^2} r \cdot r\, dy\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^2 \, [y]_{r^2}^{8-r^2} \, dr\, d\theta.$$

$$= \int_0^{2\pi}\int_0^2 r^2\left(8-r^2-r^2\right) dr\, d\theta.$$

$$= \int_0^{2\pi}\int_0^2 8r^2 - 2r^4 \, dr \, d\theta.$$

$$= \int_0^{2\pi} \left[\frac{8r^3}{3} - \frac{2r^5}{5}\right]_0^2 d\theta = \int_0^{2\pi} \frac{64}{3} - \frac{64}{5} \, d\theta$$

$$= \int_0^{2\pi} 64\left(\frac{2}{15}\right) d\theta = \frac{128}{15}\left[\theta\right]_0^{2\pi} = \frac{256\pi}{15}$$

$V = \iint_R$ — 3-D.

$A = \iint 1$ — 2-D.

Double

$$\boxed{V = \iiint_T 1 \, dV.}$$

Triple Integration.

$V = L \times W \times \underbrace{H}_{= 1.} = \underbrace{L \times W}_{A.}$

Ex: Find the volume of 'T', $x = 4 - y^2$, $x + z = 4$, $x = 0$, $z = 0$.

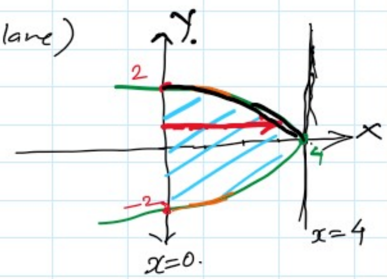
Sol^n R-3 Z-simple.

$x + z = 4$

$z = 4 - x$

$$0 \le z \le 4 - x.$$

R-2 'R' (In xy plane)



$x + z = 4 \xrightarrow{\text{xy plane}}$

$z = 0.$

$x + 0 = 4$

$x = 4$

$\rightarrow x = 0$, $\underbrace{x = 4 - y^2}_{x = -y^2 + 4} \xrightarrow{x=0} 0 = 4 - y^2$

$y^2 = 4$

$y = \pm 2$

$0 \le x \le 4 - y^2$

$-2 \le y \le 2$

$$V = \int_{y=-2}^{2}\int_{x=0}^{4-y^2}\int_{z=0}^{4-x} 1. \, dz \, dx \, dy$$

$$\int_{}^{2}\int_{}^{x=4-y^2} \left[z\right]^{4-x} dx \, dy$$

$$= \int_{-2}^{2} \int_{x=0}^{x=4-y^2} [z]_0^{4-x} \, dx \, dy$$

$$= \int_{-2}^{2} \int_{x=0}^{4-y^2} 4 - x \quad dx \quad dy$$

$$= \int_{-2}^{2} \left[4x - \frac{x^2}{2} \right]_{x=0}^{4-y^2} dy = \int_{-2}^{2} 4(4-y^2) - \frac{(4-y^2)^2}{2} dy$$

$$= \int_{-2}^{2} 16 - 4y^2 - \frac{(16 - 8y^2 + y^4)}{2} \, dy$$

$$= \int_{-2}^{2} 16 \cancel{- 4y^2} - 8 \cancel{+ 4y^2} - \frac{1}{2} y^4 \, dy$$

$$\frac{128}{5}$$

$$25.6$$

$$= \int_{-2}^{2} 8 - \frac{1}{2} y^4 \, dy$$

$$= \left[8y - \frac{1}{10} y^5 \right]_{-2}^{2}$$

$$= \left[16 - \frac{32}{10} - \left(-16 + \frac{32}{10} \right) \right]$$

$$= 32 - \frac{64}{10} = \frac{320 - 64}{10} = \frac{256}{10} = \frac{128}{5} = 25.6$$

Center of mass (COM)

$$M = \iiint_T \rho(x,y,z) \, dV$$

First moment of Mass:-

About yz-plane, $M_{yz} = \iiint_T x \, \rho(x,y,z) \, dV$

xz-plane, $M_{xz} = \iiint_T y \, \rho(x,y,z) \, dV$

xy-plane, $M_{xy} = \iiint_T z \, \rho(x,y,z) \, dV$

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

$$COM = (\bar{x}, \bar{y}, \bar{z})$$

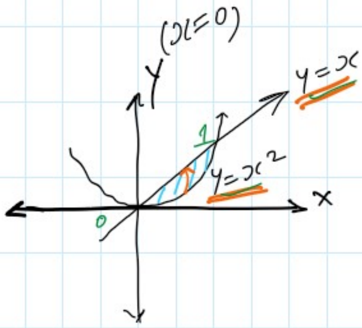
Second moment of Inertia:-

About x-axis, $I_x = \iiint_T (y^2 + z^2) \cdot \rho(x,y,z) \, dV$

y-axis, $I_y = \iiint_T (x^2 + z^2) \cdot \rho(x,y,z) \, dV$

z-axis, $I_z = \iiint_T (x^2 + y^2) \cdot \rho(x,y,z) \, dV$

Ex $\rho(x,y,z) = z$, find moment of Inertia about y-axis (I_y) for 'T', $y = x^2, y = x,$ $z = 0$, $z = x$



$x = x^2$
$x=0,\ x=1$

Sol^n:

$$I_y = \int_{x=0}^{1}\int_{y=x^2}^{x}\int_{z=0}^{x} (x^2+z^2)\cdot z \; dz\,dy\,dx.$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x}\int_{z=0}^{x} x^2\cdot z + z^3 \; dz\,dy\,dx.$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} \left[\frac{x^2 z^2}{2} + \frac{z^4}{4}\right]_{z=0}^{z=x} dy\,dx.$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} \left(\frac{x^4}{2} + \frac{x^4}{4}\right) dy\,dx.$$

$$= \int_{x=0}^{1} \frac{3x^4}{4}\cdot [y]_{x^2}^{x}\,dx.$$

$$= \int_{x=0}^{x=1} \frac{3x^4}{4}(x - x^2)\,dx.$$

$$= \int_0^1 \frac{3}{4}x^5 - \frac{3}{4}x^6\,dx$$

$$= \left[\frac{3}{4}\times\frac{1}{6}x^6 - \frac{3}{4}\times\frac{1}{7}\times x^7\right]_0^1$$

$$= \frac{3}{4}\times\frac{1}{6} - \frac{3}{4}\times\frac{1}{7} - (0)$$

$$= \frac{1}{56}$$

$z=0$, $z=x$.

on xy-plane $\downarrow z=0$
$x=0$

$y=x^2$, $y=x$.

$x^2 \le y \le x$
$0 \le x \le 1$