Triple integral

Monday, July 27, 2020 11:08 AM

Notes! 1) For $\iint_R$ 1st step was to define 'R' between two curves.

So, for $\iiint_T$ 1st step is to define 'T' between two surfaces.

→ That take care first Integral

and left with $\iint_R$; we know how to solve $\iint_R$.

dz dy dx

But 'R' could be on XY, YZ or XZ-plane.

Two Region

1) A region in R-3.
2) Then, a very specific R-2 region on a coordinate plane.

* $\iiint_T f(x,y,z)\, dV$ can be evaluated in any order dx, dy, dz

* x/y/z limit of integration [x/y/z-simple]

1) **Z-simple:** → Define R-3 region (T) between two '$z = f(x,y)$' functions.
   → 'R' will be on XY plane

2) **Y-simple:** Define R-3 region (T) between two "$y = f(x,z)$"
   'R' on XZ-plane.

3) **X-simple:** Define R-3 region (T) between two "$x = f(y,z)$" functions.
   & 'R' will be on YZ-plane.

Ex:
$$= \int_{x=0}^{2} \int_{z=0}^{z=\pi/2} \int_{y=0}^{y=3} x y^2 \cos z \, dy\, dz\, dx.$$

$$= \int_{x=0}^{x=2} \int_{z=0}^{z=\pi/2} x\left[\frac{y^3}{3}\right]_0^3 \cos z \, dz\, dx.$$

$$= \int_{x=0}^{2} \int_{z=0}^{\pi/2} 9x \cos z \, dz\, dx.$$

$$= \int^{2} 9x \left[\sin z\right]_0^{\pi/2} dx$$

$$= \int_{0}^{2} 9x \left[\sin z\right]_0^{\pi/2} \, dx$$

$$= \int_{0}^{2} 9x \, dx = \frac{9}{2}\left[x^2\right]_0^2 = 18$$

Ex

$$\int_{y=0}^{4}\int_{x=0}^{1}\int_{z=0}^{x} 2\sqrt{y}\, e^{-x^2} \, dz\, dx\, dy$$

$$= \int_{y=0}^{4}\int_{x=0}^{1} 2\sqrt{y} \quad e^{-x^2} \quad [z]_0^x \, dx\, dy$$

$$= \int_{y=0}^{y=4}\int_{x=0}^{x=1} 2x\sqrt{y} \quad e^{-x^2} \, dx\, dy$$

$u = -x^2$ | $x = 1 \to u = -1$

$du = -2x\, dx.$ | $x = 0 \to u = 0$

$\frac{du}{-2} = x\, dx$

$$= \int_{y=0}^{4}\int_{u=0}^{u=-1} -2\sqrt{y} \quad e^{u} \frac{du}{2} \, dy = \int_{y=0}^{4}\int_{u=-1}^{u=0} \sqrt{y} \; e^{u} \, du\, dy$$

$$= \int_{y=0}^{4} \sqrt{y} \quad \left[e^{u}\right]_{u=-1}^{u=0} dy = \int_{0}^{4} \sqrt{y} \, (1-e^{-1})\, dy$$

$$= \frac{2}{3}\left[y^{3/2}\right]_0^4 (1-e^{-1})$$

$$= \frac{16}{3}(1-e^{-1})$$ .ANS

Ex $f(x,y,z) = x$ and T the region bound by $x=0,\ y=0,\ z=0 \ \&\ x+y+z=1.$

Sol^n ① Z-simple :-

$x+y+z=1$ $\quad 0 \le z \le 1-x-y$

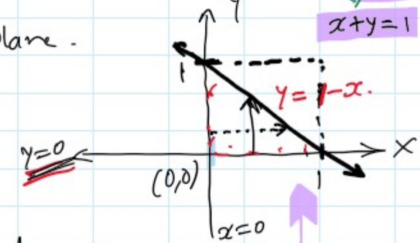
$\Rightarrow z = 1-x-y \quad , \quad z = 0$

$x+y+z=1$

$z=0$

$x+y=1$

② 'R' on XY-plane.

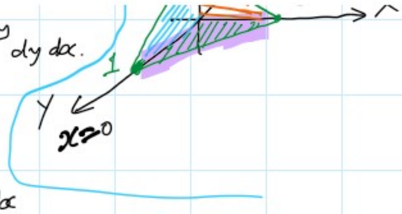


$x+y+z=1.$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} x \, dz\, dy\, dx.$$

$$= \int_{x=1}\int^{1-x} x\left[z\right]_0^{1-x-y} dy\, dx.$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{1-x} x \, [z]_0^{1-x-y} \, dy \, dx.$$



$$= \int_0^1 \int_0^{1-x} x(1-x-y) \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} (x - x^2 - xy) \, dy \, dx$$

$$= \int_0^1 \left[xy - x^2 y - \frac{1}{2} x y^2 \right]_{y=0}^{y=1-x} dx$$

$1 - 2x + x^2$

$$= \int_0^1 x(1-x) - x^2(1-x) - \frac{1}{2} x (1-x)^2 \, dx$$

$$= \int_0^1 x - x^2 - \cancel{x^2} + x^3 - \frac{1}{2} x + \cancel{x^2} - \frac{1}{2} x^3 \, dx.$$

$$= \int_0^1 \frac{1}{2} x^3 - x^2 + \frac{1}{2} x \, dx.$$

$$M = \frac{1}{24}.$$

1

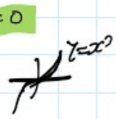
$-x^2(1-x) - \frac{1}{2} x (1-x) \, dx.$

$\int_0 \quad \frac{1}{2} x \, dx = \frac{x^4}{} \quad {}_3 \quad - \frac{1}{4} x^2 \Big]_0^1$

$\int_0^1 \frac{1}{2} \quad = \frac{1}{4} \quad \frac{2+3-8+6}{24} = \frac{1}{24}$

$\frac{1}{8} - \frac{1}{3} + \frac{1}{4}$

Ex Find mass of 'T'; $y = x^3$, $y = x$, $z = 2x$, $z = 0$
where mass density is given by



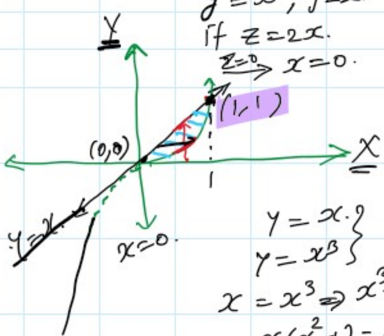
$$\rho(x, y, z) = 2z.$$

① R-3 z simple:

$z = 0$ & $z = 2x$.

$0 \le z \le 2x$

$y = x^3$, $y = x$.
If $z = 2x$.
$\xRightarrow{z=0} x = 0$.



$\left.\begin{array}{l} Y = x. \\ Y = x^3 \end{array}\right\}$

$x = x^3 \Rightarrow x^3 - x = 0$

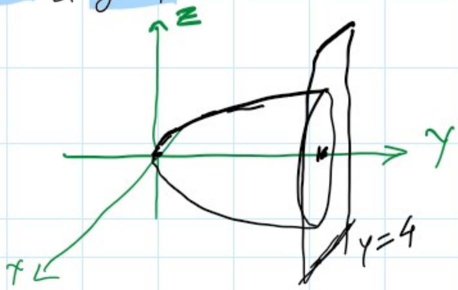
$x(x^2 - 1) = 0$

$x = 0$ or $x = 1$

$$M = \int_{x=0}^{x=1} \int_{y=x^3}^{y=x} \int_{z=0}^{2x} 2z \, dz \, dy \, dx.$$

Ex:- $\rho(x, y, z) = y$ & 'T' region bound by
$y = x^2 + z^2$ & $y = 4$

Soln :-



1) R-3 - y-simple.

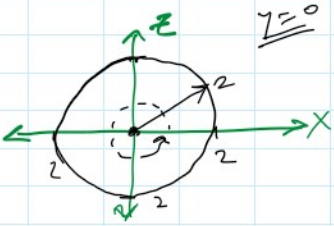
$$x^2+z^2 \le y \le 4$$

2) 'R' on xz-plane

$Y = 4$, $y = x^2+z^2$

$x^2+z^2=0$

$y=0$



$$x^2+z^2=4$$

$$r^2=4 \Rightarrow r = 2$$

$$0 \le r \le 2$$

$$0 \le \theta \le 2\pi$$

$$M = \int_{x=0}^{x=2\pi} \int_{r=0}^{2} \left(\int_{y=x^2+z^2}^{y=4} y \, dy \right) (r \, dr \, d\theta)$$

$x^2+z^2 = r^2$ ← imp.

ANS :- $\frac{64\pi}{3}$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \left[\frac{1}{2} y^2 \right]_{y=r^2}^{y=4} r \, dr \, d\theta$$

$$= \int_{0}^{2\pi} \int_{r=0}^{2} \left[8 - \frac{1}{2} r^4 \right] r \, dr \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} 8r - \frac{1}{2} r^5 \, dr \, d\theta$$

$$= \int_{0}^{2\pi} \left[4r^2 - \frac{1}{12} r^6 \right]_{r=0}^{r=2} d\theta = \int_{0}^{2\pi} 16 - \frac{16}{3} \, d\theta$$

$$M = \left[\frac{32}{3} \theta \right]_0^{2\pi} = \frac{64\pi}{3}.$$

HW Q. $\rho(x,y,z) = z$ & T region bound by

$$x^2+z^2=4, \quad x=2y, \quad y=0, \quad z=0, \quad x \ge 0.$$

Hint: ⇒ Solve by z-simple and y-simple both.