Everything Polynomic

Friday, July 10, 2020 6:06 PM

Power Series:

Geometric Series :-

$$\left(\frac{1}{1-x}\right) = 1+x+x^2+x^3+x^4+ \cdots + x^n + \cdots \infty$$

common ratio x $\quad \sum_{n=0}^{\infty} x^n$

$|x| < 1 \Leftarrow -1 \le x \le 1$

$S_\infty = \frac{1}{1-x}$ (sum to infinite)

Definition:-

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n + \cdots$$

where $C_n \in R$ is a power series centred at $\underline{\underline{x=0}}$

$]-R, R[$ where R is radius of converge.

a) Find a power series $y = \frac{1}{1+x}$ on the interval $]-1, 1[$

Soln

$y = \frac{1}{1-(-x)}$ $\quad |-x| < 1$ (for convergence)

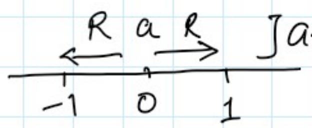
$\boxed{|x| < 1}$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x)^1 + (-x)^2 + (-x)^3 + \cdots$$

$$= 1 - x + x^2 - x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

$]-1, 1[\leftarrow$ interval of convergence



$]a-R, a+R[$

Radius of convergence is R.

$\boxed{R = 1.}$

b) Find a power series that represents the f^n

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$y = \frac{1}{1-2x}$ on the interval $]-\frac{1}{2}, \frac{1}{2}[$.

$y = \frac{1}{1-2x}$.

$|2x| < 1$

$|x| < \frac{1}{2}$

$R = \frac{1}{2}$

$r = 2x$.

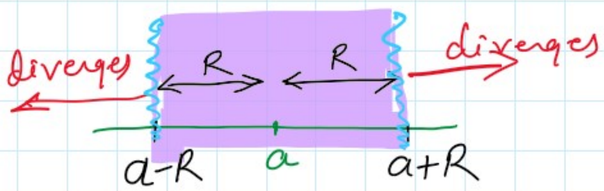
$$\frac{1}{1-2x} = 1 + 2x + (2x)^2 + \cdots + (2x)^n + \cdots$$
$$= 1 + 2x + 4x^2 + \cdots$$

Def^n $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$

power series centered at $x=a$

radius of convergence as R

interval of convergence $]a-R, a+R[$



Q Find a power series to represent $f(x) = \frac{1}{x}$.
stating the interval & radius of convergence.

$$f(x) = \frac{1}{1-(1-x)} = \frac{1}{1+(x-1)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

common ratio $= -(x-1)$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n(x-1)^n + \cdots$$

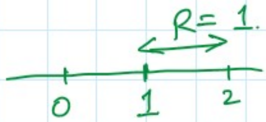
$|-(x-1)| < 1$.

$|x-1| < 1$.

$\Rightarrow -1 < x-1 < 1$

$0 < x < 2$ $]0, 2[$

Centered at $x = 1$.



b) Find a power series to represent $f(x) = \frac{1}{2+x}$ & state the interval & radius of convergence.

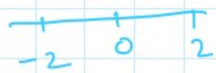
Soln:

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2}\left[\frac{1}{1-\left(-\frac{x}{2}\right)}\right] \quad ; \quad \left|\frac{-x}{2}\right| < 1.$$

$$|x| < 2$$

$$-2 < x < 2$$

R = 2



$$= \frac{1}{2}\left[1 + \left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2 + \cdots\right]$$

$$\frac{1}{1-(-1-x)} = \frac{1}{1-(-x-1)}$$

$$CR = |-x-1| < 1.$$

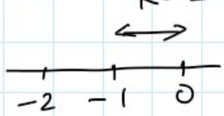
$$|x+1| < 1.$$

$$-1 < x+1 < 1.$$

$$-1-1 < x+1-1 < 1-1$$

$$-2 < x < 0$$

$x = -1$.



$$\frac{1}{1-(-x-1)} = 1 + (-x-1) + (-x-1)^2 + \cdots$$

Ratio :- Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

converges (absolutely) If $L < 1$.

& diverges otherwise.

Ex Find the interval of convergence of the infinite series.

$$\frac{(x+2)}{3\times 1} + \frac{(x+2)^2}{3^2\times 2} + \frac{(x+2)^3}{3^3\times 3} + \cdots$$

Soln

$$u_n = \frac{(x+2)^n}{3^n \times n}$$

$$\lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{3^{n+1}\times(n+1)} \times \frac{3^n\times n}{(x+2)^n}\right| \quad \text{— by ratio Test}$$

$$= \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(x+2)^n} \times \frac{\cancel{3^n}\times n}{\cancel{3^n}\cdot 3(n+1)}\right|$$

$$= \lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{(x+2)^n} \times \frac{\cancel{3^n} \times n}{\cancel{3^n} \cdot 3(n+1)} \right|$$

$$= \lim_{n\to\infty} \left| (x+2) \times \left(\frac{n}{3n+3}\right) \right| = \left| \frac{x+2}{3} \right| < 1$$

$L = \frac{1}{3}$

$R = \frac{1}{L} = 3$

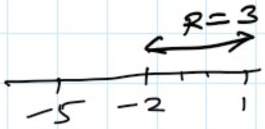
$|x+2| < 3$

$|x+2| < 3$

$-3 < x+2 < 3$

$-3-2 < x+2-2 < 3-2$

$-5 < x < 1$



$$u_n = \frac{(x+2)^n}{3^n \cdot n}$$

For $x = 1$, $u_n = \frac{(1+2)^n}{3^n \cdot n} = \frac{1}{n}$

$\sum u_n = \frac{1}{n}$ is divergent

For $x = -5$, $u_n = \frac{(-5+2)^n}{3^n \cdot n} = \frac{(-3)^n}{3^n \cdot n} = \frac{(-1)^n}{n}$

Alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n \longrightarrow$ converges.

(1) $0 \le b_{n+1} \le b_n$

(2) $\lim_{n\to\infty} b_n = 0$

Interval of converges.

$-5 \le x < 1$

$R = 3$

$C = -2$

Ex Consider the power series $\sum_{k=1}^{\infty} k\left(\frac{x}{2}\right)^k$

i) Find radius of convergence

ii) Interval of convergence

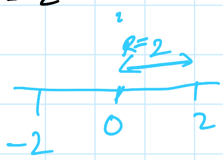
1) $\lim_{n\to\infty} \left| \frac{(k+1)\left(\frac{x}{2}\right)^{k+1}}{k\left(\frac{x}{2}\right)^k} \right| = \lim_{n\to\infty} \left| \left(\frac{x}{2}\right) \cdot \frac{k+1}{k} \right|$

1) $\lim_{n\to\infty} \left| \dfrac{\cdots}{K \left(\frac{x}{2}\right)^K} \right| = \lim_{n\to\infty} \left| \left(\frac{x}{2}\right) \cdot \dfrac{\cdots}{K} \right|$

$= \left|\frac{x}{2}\right| (1)$ $\qquad R = \dfrac{1}{\frac{1}{2}} = 2$

2) Interval of convergence. $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

$-2 < x < 2 \qquad]-2, 2[$



For Boundary condition:-

$\underline{\underline{x=2}}$ $\qquad \sum_{K=1}^{\infty} K\left(\frac{2}{2}\right)^K = \sum_{K=1}^{\infty} K$ —— diverges

$\underline{\underline{x=-2}}$ $\qquad \sum_{K=1}^{\infty} K\left(\frac{-2}{2}\right)^K = \sum_{K=1}^{\infty} K(-1)^K$

$\lim_{K\to\infty} K(-1)^K = \underline{\infty} \rightarrow$ diverges.

Interval of convergence.

$-2 < x < 2$

$R = 2$

$C = 0$

$\underline{\underline{Ex}}$ Find the radius of convergence of the series $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^n}{(n+1)3^n}$

$\lim_{n\to\infty} \left| \dfrac{(-1)^{n+1} x^{n+1}}{(n+2)\, 3^{n+1}} \div \dfrac{(-1)^n x^n}{(n+1)\, 3^n} \right|$

$= \lim_{n\to\infty} \left| \dfrac{(-1)^{n+1} x^{n+1}}{(n+2)\, 3^{n+1}} \times \dfrac{(n+1)\, 3^n}{(-1)^n x^n} \right|$

$= \lim_{n\to\infty} \left| \dfrac{-1 \; x(n+1)}{3(n+2)} \right| = \lim_{n\to\infty} |x| \dfrac{(n+1)}{3(n+2)}$

$= \dfrac{|x|}{3}$

$R = \dfrac{1}{L} = 3$

$\dfrac{|x|}{3} < 1$

$$-1 < \frac{x}{3} < 1$$
$$-3 < x < 3 \quad (IOC)$$

$\underline{x=-3}$,
$$\sum \frac{(-1)^n \cdot (-3)^n}{(n+1)\,3^n}$$
$$= \sum \frac{(-1)^n \cdot (-1)^n}{(n+1)} \cdot \frac{3^n}{3^n}$$
$$= \sum \frac{(-1)^{2n}}{n+1}$$
$$= \sum \left(\frac{1}{n+1}\right) \longleftarrow$$ General Harmonic Series.

$\downarrow$ diverges.

$\underline{x=3}$;
$$\sum \frac{(-1)^n (3)^n}{(n+1)\,3^n}$$
$$= \sum \frac{(-1)^n}{n+1}$$ —— converges. AHS.

$$-3 < x \leq 3 \quad (\text{Interval of converges})$$

Introduction to Taylor :-

$$f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + C_4(x-a)^4 + \cdots$$

Differentiate both side w.r.t. x.

$$f'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + 4C_4(x-a)^3 + \cdots$$
$$\xrightarrow{D} f''(x) = 2C_2 + 3\cdot 2\,C_3(x-a) + 4\cdot 3\,C_4(x-a)^2 + \cdots$$
$$\xrightarrow{D} f'''(x) = 3\cdot 2\,C_3 + 4\cdot 3\cdot 2\,C_4(x-a) + \cdots$$

$x=a \longrightarrow$ $f'(a) = C_1$, $f'''(a) = 3\cdot 2\cdot C_3$

$f''(a) = 2C_2$, $C_3 = \dfrac{f'''(a)}{3\cdot 2\cdot 1}$

$$C_2 = \frac{f''(a)}{2\cdot 1}$$

$$C_4 = \frac{f^{iv}(a)}{4!}$$

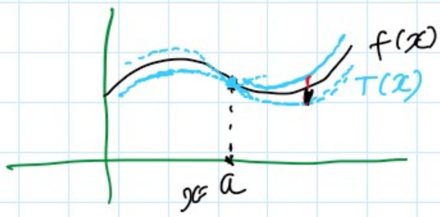
plug $x=a$ in $f(x)$

$f(a) = C_0$

$$C_n = \frac{f^n(a)}{n!}$$

$$T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{\cdot}(x-a) + \frac{f'''(a)}{3!}(x-a) +$$

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a) + \frac{f'''(a)}{3!}(x-a) + \cdots + \frac{f^{n}(a)}{n!}(x-a)$$



|error|

$$f(x) = f(a) + f'(a)(x-a) + \frac{f'(a)}{2!}(x-a) + \cdots + \frac{f^{n}(a)}{n!}(x-a)^n + R_n(x)$$

$R_n(x)$ is remainder (or error)

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

$$(a-r) — a — (a+r)$$

$$c \in (a-r, a+r)$$

Ex: compute the 4th order Taylor polynomial for the function $f(x) = \sin x$ at $x = \frac{\pi}{4}$

Sol^n

$$\left.\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{iv}(x) &= \sin x \end{aligned}\right\} \xrightarrow{x=\frac{\pi}{4}} \begin{aligned} f\left(\tfrac{\pi}{4}\right) &= \tfrac{\sqrt{2}}{2} \\ f'\left(\tfrac{\pi}{4}\right) &= \tfrac{\sqrt{2}}{2} \\ f''\left(\tfrac{\pi}{4}\right) &= -\tfrac{\sqrt{2}}{2} \\ f'''\left(\tfrac{\pi}{4}\right) &= -\tfrac{\sqrt{2}}{2} \\ f^{iv}\left(\tfrac{\pi}{4}\right) &= \tfrac{\sqrt{2}}{2} \end{aligned}$$

$$T_4(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{2\cdot 2!}\left(x-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2\cdot 3!}\left(x-\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2\cdot 4!}\left(x-\frac{\pi}{4}\right)^4$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Maclaurin series:

If $a=0$, then taylor series is also referred as the mclauren series.

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \cdots$$

a) Find the mclaurine series generated by $f(x) = e^x$, & find its radius of convergence.

Sol^n

$f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$

$f^n(x) = e^x$.

for $x=0$, $f^n(0) = e^0 = 1$.

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{K=0}^{\infty} \frac{x^K}{K!}$$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!} \div \frac{x^n}{n!}\right| = |x| \lim_{n\to\infty}\frac{1}{n+1} = 0<1$$

$$R = \frac{1}{0} \to \infty$$

Ex Find Mclaurin Series for $f(x) = \ln(x+1)$ & determine its radius of convergence.

Useful application of power series:

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$$

$$= \lim_{x\to 0} \frac{\left(\cancel{1} - \cancel{\frac{x^2}{2!}} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots\right) - \cancel{1} + \cancel{\frac{x^2}{2}}}{x^4}$$

$$= \lim_{x\to 0} \frac{\frac{\cancel{x^4}}{4!} + \text{x containg term}}{\cancel{x^4}}$$

$$= \lim_{x \to 0} \frac{\frac{1}{4!}\cancel{x^4}}{\cancel{x^4}}$$

$$= \frac{1}{24}$$

Ex $\lim_{x \to 0} \frac{\cos(x^4) - 1 + \frac{1}{2}x^8}{x^{16}}$ $\left(\text{ANS} = \frac{1}{4!}\right)$

Ex Compute the Mclaurine series of the following

1) $\left(\frac{\sin(x)}{x}\right) = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}{x^1}$ ← By Mclaurine

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

2) $\frac{\sin(x^2)}{x^2} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}}{x^2}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n+1)!}$$

s^{4n}

3) $\int_0^x \frac{\sin(s^2)}{s^2}\, ds = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n (s^2)^{2n}}{(2n+1)!}\, ds$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left. \frac{s^{4n+1}}{(4n+1)} \right|_0^x$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{(x)^{4n+1}}{(4n+1)}$$

Using Taylor Series to approximate f^n:-

1) Approximate $\cos(0.1)$ with error less than 10^{-20}.

Soln: $a = 0$

$$f(x) = \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

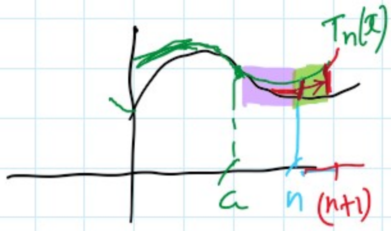
$$f(x) = \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$\underline{x = 0.1}$

$$\cos(0.1) = \sum_{n=0}^{\infty} (-1)^n \frac{(0.1)^{2n}}{(2n)!} < u_{n+1}$$

$$u_{n+1} < 10^{-20}$$

$$\frac{(0.1)^{2n}}{(2n)!} < 10^{-20}$$



$n = 5$ (By GDC)

$$\cos(0.1) = \sum_{n=0}^{5} \frac{(-1)^n (0.1)^{2n}}{(2n)!} = 0.99995$$

Ex find the McLaurine series for $f(x) = \sin x$ & prove that it represents the f^n for all x.

Soln:-

$$\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots$$

$$\left[\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)!} \right.$$

According to taylor.

$0 < t < x$

$$R_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!} x^{n+1}$$

$f(x) = \sin x.$

$$|f^{(n+1)}(t)| \le 1$$

$$0 < \frac{|f^{(n+1)}(t)|}{(n+1)!} |x^{n+1}| < \left(\frac{|x|^{n+1}}{(n+1)!} \right)$$

By squeeze theorem

$$\frac{1}{\infty} = 0$$

$$\therefore \left(\lim_{n \to \infty} \frac{|x|^{(n+1)}}{(n+1)!} = 0 \right)$$

By squeeze theorem $\quad \frac{\;}{\infty} = 0$

$$\therefore \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

$$\lim_{n\to\infty} R_n(x) = 0.$$

Hence.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \text{ for all } x.$$