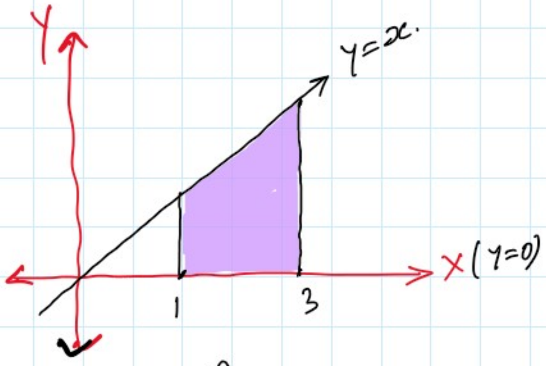
Center of Mass and Moments of Inertia

Wednesday, July 22, 2020 10:58 AM

One revision problem:- Ans. $2\ln(\sqrt{2}+1)$

Ex $\displaystyle\int_{x=1}^{x=3}\int_{y=0}^{y=x} \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx.$



$\displaystyle\int_{\theta=0}^{\theta=\pi/2}\int_{r=\sqrt{2}}^{r=3\sqrt{2}} \frac{1}{r}\cdot r\,dr\,d\theta.$

$y=0$, $y=x$.

$\theta=0$, $\theta=\tan^{-1}\left(\frac{y}{x}\right) = \pi/4$

$x=1$, $x=3$

$r\cos\theta=1$ | $r\cos\theta=3$

$r=\frac{1}{\cos\theta}$

$\Rightarrow r= \underset{LL}{\sec\theta}$ | $r=\underset{UL}{3\sec\theta}$

$$V=\int_{\theta=0}^{\theta=\pi/4}\int_{r=\sec\theta}^{r=3\sec\theta}\frac{1}{r}\cdot r\,dr\,d\theta$$

$$=\int_{\theta=0}^{\theta=\pi/4}\int_{r=\sec\theta}^{r=3\sec\theta} 1\,dr\,d\theta.$$

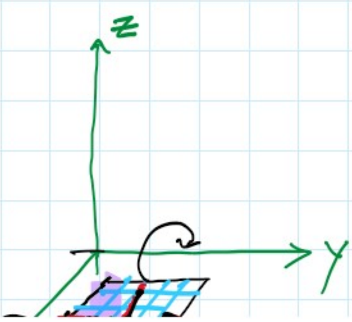
$$=\int_0^{\pi/4}\left[r\right]_{\sec\theta}^{3\sec\theta}d\theta = \int_0^{\pi/4} 2\sec\theta\,d\theta$$

$$=2\left[\ln|\sec\theta+\tan\theta|\right]_0^{\pi/4} = 2\ln(\sqrt{2}+1)$$

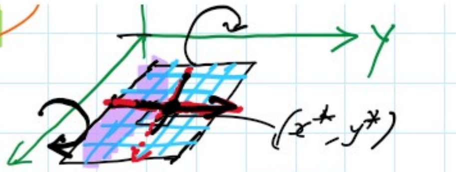
If the lamina has a mass density of $\rho(x,y)$ at any point (x,y) then the mass of the plate is

$M=$ mass density $\times$ Area.

$$M=\sum_{i=1}^{\infty}\sum_{j=1}^{n}\rho(x^*,y^*)\,\Delta x\,\Delta y$$



$$M = \sum_{i=1}\sum_{j=1} \rho(x^*, y^*)\, \Delta x \Delta y$$



$$M = \iint_R \rho(x,y)\, dx\, dy \qquad \times \qquad M = \sigma(x^*, y^*)\, \Delta x\, \Delta y$$

moment of mass $=$ distance $\times$ mass

$= Y \times M$

First moments of Mass:-

Tendency for the lamina to rotate about an axis.

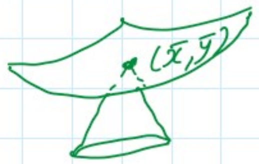
$$\underline{M_x} = \iint_R y\, \rho(x,y)\, dA \longleftarrow \sum\sum y_B^*\, \rho(x^*, y^*)\, \Delta x \Delta y$$

$$M_y = \iint_R x \cdot \rho(x,y)\, dA$$

center of mass :-

x-co-ordinate; $\bar{x} = \dfrac{M_y}{M}$

y-co-ordinate; $\bar{y} = \dfrac{M_x}{M}$



Ex Let R be the unit square

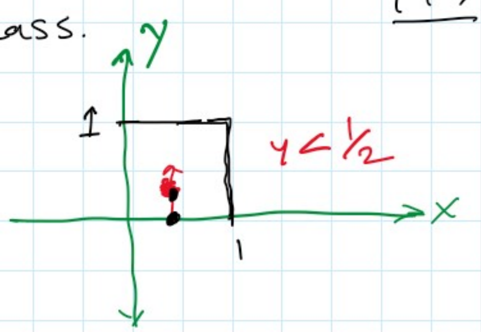
$R = \{(x,y);\ 0 \le x \le 1,\ 0 \le y \le 1\}$ suppose the density of R is given by the f^n

$$\rho(x,y) = \frac{1}{y+1}.$$

find center of mass.

$$\underline{\rho(x,y)} = \frac{1}{y+1}$$

Soln:



$$M = \int_{x=0}^{x=1}\int_{y=0}^{y=1} \frac{1}{y+1}\, dy\, dx.$$

$$= \int [\ln(y+1)]_0^1\, dx = \int_0^1 \ln 2\, dx = \ln 2$$

$$= \int_0^1 [\ln(y+1)]_0^1 \, dx = \int_0^1 \ln 2 \, dx = \ln 2$$

$$M_x = \int_{x=0}^{x=1}\int_{y=0}^{y=1} \frac{y}{y+1} \, dy\, dx = \int_0^1\int_0^1 \left(1 - \frac{1}{y+1}\right) dy\, dx = \int_0^1 [y - \ln(y+1)]_0^1 \, dx$$
$$= 1 - \ln 2$$

$$M_y = \int_{x=0}^{x=1}\int_{y=0}^{y=1} \frac{x}{y+1} \, dy\, dx = \int_0^1 x \ln 2 \, dx = \left[\frac{x^2}{2}\ln 2\right]_0^1$$
$$= \frac{1}{2}\ln 2$$

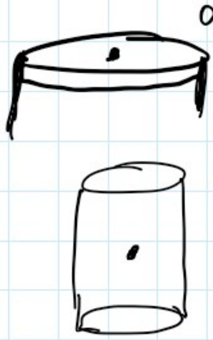
$$\text{Center of mass} = (\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{\frac{1}{2}\ln 2}{\ln 2}, \frac{1-\ln 2}{\ln 2}\right)$$
$$= \left(\frac{1}{2}, 0.44\right)$$

Second moment of Inertia:-

Moment of Inertia about 'x' axis.

$$I_x = \iint_R y^2 \rho(x,y)\, dA$$

disk



About Y-axis:

$$I_y = \iint_R x^2 \rho(x,y)\, dA.$$

About origin:

$$I_0 = I_x + I_y$$

Radius of Gyration:-

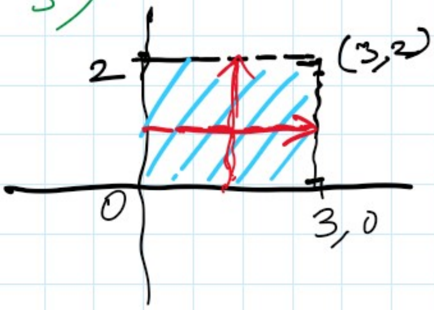
about Y axis $\rightarrow$ $\bar{\bar{x}} = \sqrt{\frac{I_y}{M}}$

about X-axis. $\rightarrow$ $\bar{\bar{y}} = \sqrt{\frac{I_x}{M}}$

Ex:/ Find center of mass & radius of gyration

Ex: Find center of mass & radius of gyration for a lamina of a mass density of $\rho(x,y) = y$ and occupying the region $(0,0), (3,0), (3,2), (0,2)$.

Center of mass $\left(\frac{3}{2}, \frac{4}{3}\right)$

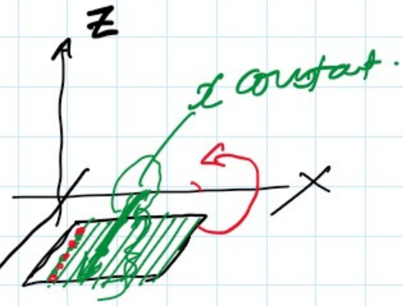


$$\text{center of mass} = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

$$M = \int_{y=0}^{y=2}\int_{x=0}^{x=3} y \, dx\, dy$$

$$M = \int_0^2 y\,[x]_0^3\, dy = \int_0^2 3y\, dy = \left[\frac{3y^2}{2}\right]_0^2 = 6$$

$$M_x = \int_0^2\int_0^3 y \cdot y \, dx\, dy =$$



moment of mass about x-axis.
= distance × $\rho(x,y)$

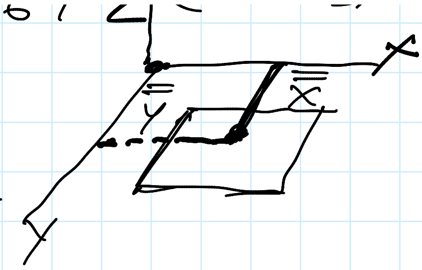
$\sum\sum y^* \rho(x^*, y^*) \Delta x \Delta y$

$M_x = 8$

$$M_y = \int_0^2\int_0^3 x\,y \, dx\, dy = 9$$

$$\text{Center of mass} = \left(\frac{9}{6}, \frac{8}{6}\right) = \left(\frac{3}{2}, \frac{4}{3}\right)$$

2 3



$$I_x = \int_0^2 \int_0^3 y^2 \cdot \rho(x,y)\, dx\, dy$$

$$= \int_0^2 \int_0^3 y^2 \cdot y \cdot dx\, dy = \int_0^2 y^3 [x]_0^3\, dy$$

$$= \int_0^2 3y^3\, dy = \left[\frac{3y^4}{4}\right]_0^2 = 12$$

$$I_y = \int_0^2 \int_0^3 x^2 y\, dx\, dy = 18$$

$$\left.\begin{aligned} \bar{\bar{X}} &= \sqrt{\frac{I_y}{M}} = \sqrt{\frac{18}{6}} = \sqrt{3} \\ \bar{\bar{Y}} &= \sqrt{\frac{I_x}{M}} = \sqrt{\frac{12}{6}} = \sqrt{2} \end{aligned}\right\} \text{Radii of gyration.}$$