Ratio test for convergence.

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Ex a) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$a_{n+1} < a_n$

$\left(\frac{a_{n+1}}{a_n}\right) < 1$

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}$$

$$= \lim_{n\to\infty} \frac{n+1}{n} \times \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{1}\right) \times \frac{\cancel{2^n}}{\cancel{2^n}\cdot 2}$$

0

$$= \frac{1}{2} < 1.$$

The series is converges.

b) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

$$\lim_{n\to\infty} \frac{2^{n+1} \times n!}{(n+1)! \times 2^n} = 0$$

$L < 1$ — converges.

c) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

convergent
According to p-series

d) $\sum_{n=1}^{\infty} \frac{1}{n}$

Divergent

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

$L = 1$. Inconclusive.
(Ratio test)

Absolute convergence of series

$1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \cdots$

$\sum a_n$

$\sum |a_n| \longrightarrow$ Converges

$\sum a_n \longrightarrow$ converges

$-|a_n| \le a_n \le |a_n|$

$\cancel{-|a_n| + |a_n|} \le a_n + |a_n| \le |a_n| + |a_n|$

$0 < (a_n + |a_n|) \le 2|a_n|$

(b_n)

If $\sum |a_n|$ converges

$\rightarrow \underbrace{\sum 2|a_n|}_{b_n} \longrightarrow$ converges

Comparison Test. $a_n \le b_n$

If $b_n \to$ converges

$\Rightarrow a_n \to$ converges.

$a_n + |a_n| \to$ converges. (By Comparison Test)

$a_n = \underbrace{(a_n + |a_n|)}_{C} - \underbrace{(|a_n|)}_{C}$

$\Downarrow$

Converges.

$\sum |a_n| \longrightarrow$ converges $\Rightarrow \sum a_n \to$ converges.

a) $1 + \left(\frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + - \cdots\right)$

General term $\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n^2}\right)$

$\sum \left|(-1)^{n-1} \times \frac{1}{n^2}\right| = \sum \frac{1}{n^2} \rightarrow$ Converges (By p-Series test)

$$\left.\begin{array}{l}\sum |a_n| \rightarrow \text{converges.} \\ \sum a_n \rightarrow \text{converges.}\end{array}\right\}$$

Ex $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

$$\sum |a_n| = \sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$$

$\Downarrow$

diverges (p-series test)

$\sum a_n$ is also diverges

Hence $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \rightarrow$ diverges.

c) $\sum_{n=1}^{\infty} \left(\frac{\cos n}{n^2}\right)$

$-1 \leq \cos n \leq 1$

$\sum \left|\frac{\cos n}{n^2}\right|$

$\Rightarrow \left|\frac{\cos n}{n^2}\right| \leq \left(\frac{1}{n^2}\right)$

then

If

Converges (p-series test)

Converges (By comparison)

Hence.

$\sum \frac{\cos n}{n^2} \rightarrow$ converges (ACT)

d) $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

$u_n = \left(\frac{n^3}{3^n}\right)$

$$= \lim_{n\to\infty} \frac{(n+1)^3 \times 3^n}{3^{n+1} \times n^3}$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n}\right)^3}_{1.} \times \frac{\cancel{3^n}}{\cancel{3^n}\cdot 3} = \frac{1}{3} < 1.$$

$\sum |a_n|$ converges. $\Rightarrow \sum a_n$ also converges (By ACT)

Conditional convergence of series.

Alternative series test (Leibniz's theorem)

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots, \quad (u_n > 0) \quad \forall\ n \in \mathbb{Z}^+$$

$\Rightarrow$ converges

if
- a) $u_n \geq u_{n+1}$ for all $n \geq N \in \mathbb{Z}^+$
- b) $\lim_{n\to\infty} u_n = 0.$ ✓

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Definition: The infinite series $\sum u_n$ is conditionally convergent if $\sum u_n$ converges But $\sum |u_n|$ diverges. #

Ex AHS $\quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

i) $u_n \geq u_{n+1} \qquad \frac{1}{n+1} < \frac{1}{n}$

ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

AHS is converges coz $\sum |u_n|$ is diverges

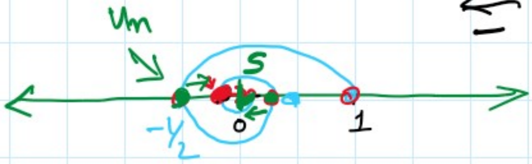
Ex
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$

a) Absolute value of term is decreasing

b) $\lim_{n\to\infty} u_n = 0$ & by comparison $\frac{1}{n} \le \frac{1}{\ln(n+1)}$

$\because$ $\frac{1}{n}$ diverges $\Rightarrow \frac{1}{\ln(1+n)} \to$ diverges

Hence $\sum \frac{(-1)^n}{\ln(n+1)}$ conditionally convergent. $\not\Rightarrow$ $\Leftarrow$ $\sum \frac{(-1)^{n+1}}{n}$



$$\sum \left(\frac{(-1)^{n+1}}{n}\right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ----$$

partial sum

$u_n \to 0$ $\qquad L \to (S)$ sum

$$|S_\infty - S_n| = |R_n| \le u_{n+1}$$

truncation error.

$$\sum_{n=1}^{100} \left(\frac{1}{n}\right) = A$$

$\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \right.$

$$\sum_{n=1}^{101} \left(\frac{1}{n}\right) = B.$$

$$A - B = \frac{1}{101}$$

$$\boxed{\begin{array}{c} S_{101} - S_{100} = \frac{\pm 1}{101} \\ |S_{101} - S_{100}| = \frac{1}{101} \end{array}}$$

Ex Approximate the sum of the series $\boxed{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n!}}$ using its first six terms.

Soln

1) $\frac{1}{n!} > \frac{1}{(n+1)!}$ for $\forall n$

2) $\lim_{n \to \infty} \frac{1}{n!} = 0$

Hence converges.

Leibniz theorem.

$u_7 = \frac{1}{7!} = \frac{1}{5040}$

$$S = S_6 + \left(\frac{1}{5040}\right)$$

$$S = \underline{\underline{0.632}}.$$

Ex. Approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3-1}$ with an error of less that 0.001.