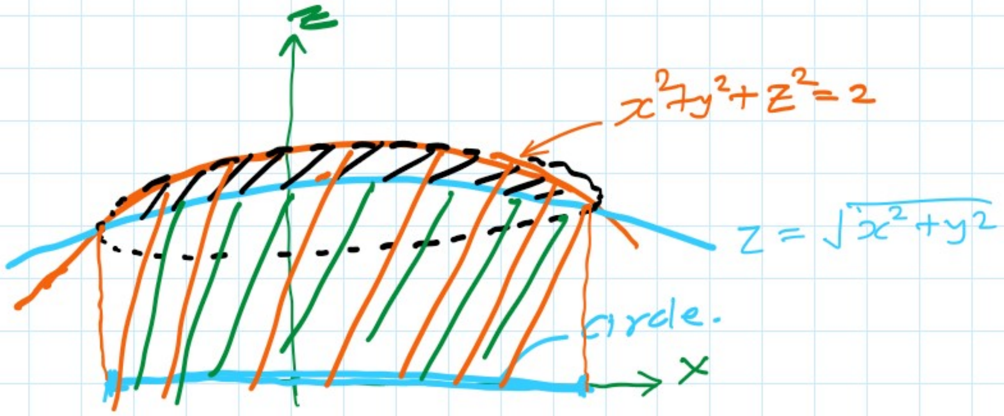
Application of double integral

Monday, July 20, 2020 10:55 AM

Ex volume between $x^2+y^2+z^2=2$ & $z=\sqrt{x^2+y^2}$



$z=\sqrt{x^2+y^2} \Rightarrow z^2=x^2+y^2$

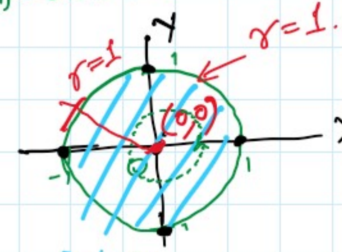
$x^2+y^2+z^2=2$ — 1st surface.

$x^2+y^2+x^2+y^2=2$

$\Rightarrow 2x^2+2y^2=2$

$\Rightarrow x^2+y^2=1.$ —

$r=1.$



Bottom: $z=\sqrt{x^2+y^2}$, Top: $x^2+y^2+z^2=2$

$z^2=2-x^2-y^2$

$z=\sqrt{2-x^2-y^2}$

$(x,y)\equiv(0,0)$

$z=0$, $z=\sqrt{2}$

Top – Bottom:-

$$f=\sqrt{2-x^2-y^2}-\sqrt{x^2+y^2}=\sqrt{2-r^2}-\sqrt{r^2}$$
$$=\sqrt{2-r^2}-r$$

Iterated Integral form:- $V=\iint_R f(x,y)\,dA$

$$V=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\left(\sqrt{2-r^2}-r\right)r\,dr\,d\theta$$

$$V=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} r\sqrt{2-r^2}-r^2\,dr\,d\theta$$

ANS:

$\frac{4\pi}{3}(\sqrt{2}-1)$

$\longrightarrow$?

$2\pi\sqrt{2}-\frac{4\pi}{3}$

$\theta=2\pi$ $r=1$

$$\theta=0 \quad r=0$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} r\sqrt{2-r^2}\,dr\,d\theta - \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} r^2\,dr\,d\theta$$

$$2\pi\sqrt{2} - \frac{4\pi}{3}$$

$$\frac{4\pi}{3}\left(\frac{3}{2}\sqrt{2} - 1\right)$$

$$2 - r^2 = u$$
$$-2r\,dr = du$$
$$r\,dr = -\frac{1}{2}\,du$$

$r=0,\ u=2$
$r=1,\ u=1$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{u=1}^{u=2} \frac{1}{2}\sqrt{u}\,du\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\left[\frac{u^{3/2}}{3}\right]_1^2 d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{2^{3/2}}{3} - \frac{1^{3/2}}{3}\,d\theta.$$

$$= \int_{\theta=0}^{\theta=2\pi}\left(\frac{2\sqrt{2}}{3} - \frac{1}{3}\right) d\theta$$

$$= \left[\frac{2\sqrt{2}-1}{3}\right]\left[\theta\right]_0^{2\pi}$$

$$= \left(\frac{2\sqrt{2}-1}{3}\right)2\pi$$

$$\int_{\theta=0}^{\theta=2\pi}\left[\frac{r^3}{3}\right]_0^1 d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\frac{1}{3}\,d\theta$$

$$= \left[\frac{\theta}{3}\right]_0^{2\pi}$$

$$= \frac{2\pi}{3}$$

$$V = \left(\frac{2\sqrt{2}-1}{3}\right)2\pi - \frac{2\pi}{3}$$

$$= \frac{4\pi\sqrt{2}}{3} - \frac{2\pi}{3} - \frac{2\pi}{3}$$

$$= \frac{4\pi\sqrt{2}}{3} - \frac{4\pi}{3}$$

$$= \frac{4\pi}{3}(\sqrt{2}-1)$$ — ANSWER.

$V = \pi r^2 h$

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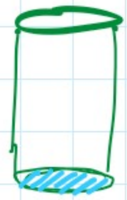
$h = 1 \Rightarrow V = \boxed{\pi r^2}$ (unit)^3



$$V = \iint_R f(x,y)\, dA,$$

$f(x,y) \to$ height.

$\Sigma\Sigma$

$$V = A \cdot H$$

$$\Rightarrow A = \frac{V}{H} \Rightarrow A = \frac{\iint_R \cancel{f(x,y)}\, dA}{\cancel{f(x,y)}}$$

$$A = \iint_R dA.$$

Ex: Find area of 'R' where R is the region bound by $r = 3\cos\theta$.

Soln

$r = 3\cos\theta$ $\qquad r^2 = x^2 + y^2$

$\Rightarrow r^2 = 3 r\cos\theta$ $\qquad x = r\cos\theta$

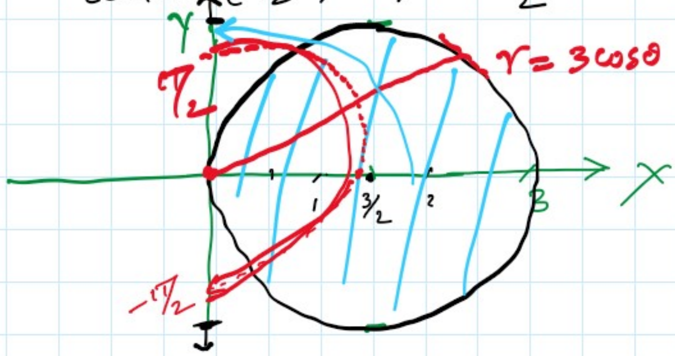
$\Rightarrow x^2 + y^2 = 3x$

$\Rightarrow x^2 - 3x + y^2 = 0$

$\Rightarrow x^2 - 3x + \frac{9}{4} + y^2 = \frac{9}{4}$

$\Rightarrow \left(x - \frac{3}{2}\right)^2 + y^2 = \left(\frac{3}{2}\right)^2$

center $(3/2, 0)$, $r = \frac{3}{2}$



×2

circular.

$$A = \iint_R 1\, dA$$

$$= \int \int 1 .\, dA = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=3\cos\theta} 1 .\, r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2}\right]_0^{3\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{9}{2} \cos^2\theta \quad d\theta$$

$$\int_{-a}^{a} \qquad 2\int_0^a$$

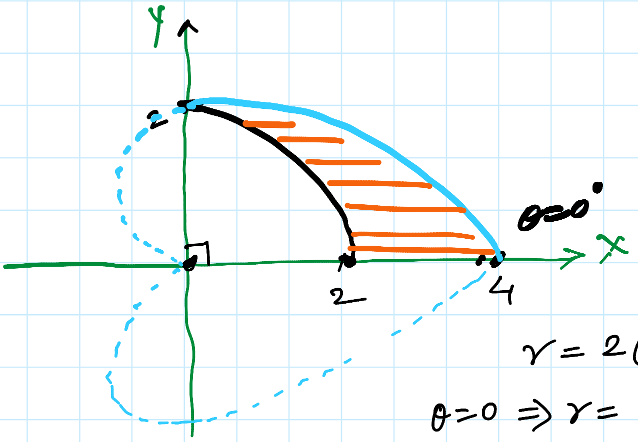
$$= 2\int_0^{\pi/2} \frac{9}{2} \cos^2\theta\, d\theta.$$

$$= \int_0^{\pi/2} 9 \cos^2\theta\, d\theta.$$

$$= \frac{9}{4}\pi$$

Ex Find the area of the region bound by $r=2$ & $r = 2(1+\cos\theta)$ on Quad. I.

Sol'n:



$r = 2(1+\cos\theta)$

$\theta = 0 \Rightarrow r = 4$

$\theta = \pi/2 \Rightarrow r = 2$

$$A = \iint_R dA = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=2}^{r=2(1+\cos\theta)} r\, dr\, d\theta$$

$\int\int_R \quad \theta=0 \quad r=2$

$$= \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_2^{2(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi/2} \frac{4(1+\cos\theta)^2}{2} - \frac{4}{2} \, d\theta$$

$$= \int_0^{\pi/2} 2(1+\cos\theta)^2 - 2 \, d\theta$$

$$= \int_0^{\pi/2} 2[1+2\cos\theta+\cos^2\theta] - 2 \, d\theta$$

$$= \int_0^{\pi/2} 4\cos\theta + 2\cos^2\theta \, d\theta$$

$$= \int_0^{\pi/2} 4\cos\theta + 1+\cos 2\theta \, d\theta$$

$$= \left[4\sin\theta + \theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \left(4 + \frac{\pi}{2}\right)$$

Ex! $\displaystyle\int_{y=-1}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \left(\frac{1}{1+x^2+y^2}\right) dx\, dy$

HW

$\frac{\pi}{2} \ln 2$

$\sin^{-1}(\sin^2\theta)$

$x=0$, $x=\sqrt{1-y^2}$

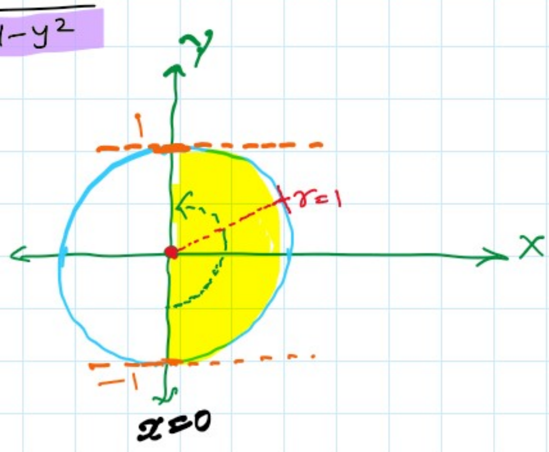
$y=-1$, $y=1$.

$x^2 = 1-y^2$

$x^2+y^2 = 1.$

$x^2 = 1-y^2$

$x = \pm\sqrt{1-y^2}$



$$\hookrightarrow x^2 = 1 - y^2$$

$$x = \pm\sqrt{1-y^2}$$

$x=0$

$$V = \int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{r=0}^{r=1} \frac{1}{1+r^2}\, r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\Big[\ln|1+r^2|\Big]_0^1 \, d\theta.$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\Big[\ln 2 - \cancelto{0}{\ln 1}\Big] \, d\theta.$$

$$= \frac{1}{2}\ln 2 \quad \Big[\theta\Big]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2}\ln 2 \quad \Big[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\Big]$$

$$= \frac{\pi}{2}\ln 2.$$