Differentiation

Thursday, June 3, 2021 5:46 AM

1) * Double angle formula:-

$\sin(2x)$, $\sin(A \pm B)$, $\cos 2x$, $\cos 4x$, $\cos(A \pm B)$

$$\left.\begin{aligned}\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\ \sin(A \pm B) &= \sin A \cos B \pm \sin B \cos A\end{aligned}\right\}$$

$$\cos(2x) = \cos(x+x)$$
$$= \cos x \cos x - \sin x \sin x.$$
$$\cos 2x = \cos^2 x - \sin^2 x \quad \text{--- (1)}$$
$$= \cos^2 x - (1 - \cos^2 x)$$
$$= \cos^2 x - 1 + \cos^2 x.$$

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x.$

$$* \ \cos 2x = 2\cos^2(x) - 1. \quad \text{--- (2)}$$

$\cos^2 x = 1 - \sin^2 x.$

from (1) $\cos 2x = 1 - \sin^2 x - \sin^2 x$

$$* \ \cos(2x) = 1 - 2\sin^2(x) \quad \text{--- (3)}$$

$\frac{2x}{2} = x$

$\sin^2 2x + \cos^2 2x = 1$

$$\left.\begin{aligned} 1 + \cos 2x &= 2\cos^2 x. \\ 1 - \cos 2x &= 2\sin^2 x.\end{aligned}\right\} (*)$$

$\boxed{\cos(4x) = 2\cos^2 2x - 1}$

Also $\cos(2x + 2x)$

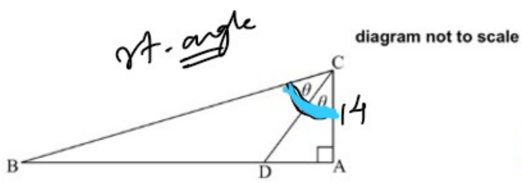
$= \cos 2x \cos 2x - \sin 2x \sin 2x$

$= \cos^2 2x - \sin^2 2x = 1 - 2\sin^2(2x)$

(c)

[Maximum mark: 7]

The following diagram shows a right triangle ABC. Point D lies on AB such that CD bisects $A\hat{C}B$.



$A\hat{C}D = \theta$ and $AC = 14\,\text{cm}$

✓ (a) Given that $\sin\theta = \frac{3}{5}$, find the value of $\cos\theta$. [3]

✓ (b) Find the value of $\cos 2\theta$. [2]

✓ (c) Hence or otherwise, find BC. [2]

$\cos C = \frac{AC}{BC}.$

$\cos 2\theta = \frac{7}{25}$

$\boxed{BC = ?}$

$\angle C = 2\theta$

$\cos 2\theta = \frac{A}{H}$

$\frac{7}{25} = \frac{14}{BC}$

$BC = 50\ \text{cm}.$

$\sin\theta = \frac{3}{5}$, $\cos\theta = ?$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\Rightarrow \cos^2\theta = 1 - \sin^2\theta$$
$$\cos^2\theta = 1 - \left(\frac{3}{5}\right)^2$$

$\Rightarrow \cos^2\theta$ ...

$\cos^2\theta = 1 - \left(\frac{3}{5}\right)^2$

$= 1 - \frac{9}{25} = \frac{16}{25}$

$\Rightarrow \cos\theta = \frac{4}{5}$ ←

b)

$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$= 2\left(\frac{4}{5}\right)^2 - 1$

$= 2 \cdot \frac{16}{25} - 1 = \frac{32-25}{25} = \frac{7}{25}$

$\cos 2\theta = \frac{7}{25}$

$\cos x$

[Maximum mark: 6]

Given that $\sin x = \frac{1}{3}$, where $0 < x < \frac{\pi}{2}$, find the value of $\cos 4x$. $= \frac{17}{81}$

$\cos 4x = \cos(2x + 2x)$

~~$\cos 4x = \frac{2\cos 2x}{x}$~~ ✗

$\cos 2x = \cos(x + x)$

$\cos 2x = 1 - 2\sin^2 x$

$= 1 - 2\left(\frac{1}{3}\right)^2 = 1 - 2 \times \frac{1}{9} = \frac{7}{9}$

(*) $\boxed{\cos 2x = \frac{7}{9}}$

$\cos(4x) = 2\cos^2(2x) - 1$

$\Rightarrow \cos 4x = 2\left(\frac{7}{9}\right)^2 - 1 = 2 \cdot \frac{49}{81} - 1$

$= \frac{98-81}{81} = \frac{17}{81}$

✓ $\boxed{\cos 4x = \frac{17}{81}}$

$\cos 8x = 2\cos^2 4x - 1$

$\cos 9x = \cos(8x + x)$

$= \cos 8x \cos x - \sin 8x \sin x$

$$\boxed{\cos 8x = 2\cos^2 4x - 1}$$

$\sin 8x.$

$$\cos 9x = \cos(8x + x)$$

$$= \cos 8x \cos x - \sin 8x \sin x$$

$$=$$

$$\sin 7x = \sin(8x - x)$$

$$= \sin 8x \cos x - \sin x \cos 8x.$$

Q

[Maximum mark: 6]

Consider the function $f(x) = x^2 e^{3x}$, $x \in \mathbb{R}$.

(a) Find $f'(x)$. [4]

(b) The graph of f has a horizontal tangent line at $x = 0$ and at $x = a$. Find a. [2]

$$y = e^{f(x)}$$

$$y' = e^{f(x)} \cdot f'(x)$$ – Chain Rule.

$$y = e^x$$

$$y' = e^x$$

$$y = u \cdot v \quad , \quad y' = uv' + vu'$$

$$\boxed{f'(x) = 3x^2 e^{3x} + 2xe^{3x}}$$

Horizontal tangent line. $f'(x) = 0$

$$x = \frac{-2}{3}$$

$$a = \frac{-2}{3}$$

[Maximum mark: 5]

Let $f(x) = \ln x - 5x$, for $x > 0$.

(a) Find $f'(x)$. $= \frac{1}{x} - 5$ [2]

(b) Find $f''(x)$. $= \frac{-1}{x^2}$ [1]

(c) Solve $f'(x) = f''(x)$. [2]

GDC

$$\frac{1}{x} - 5 = -\frac{1}{x^2}$$

$x \neq 0$

$$x - 5x^2 = -1$$

$$5x^2 - x - 1 = 0.$$

$$5x^2 - x - 1 = 0.$$

$$x = \frac{1 \pm \sqrt{21}}{10}$$

$$x = \frac{1 + \sqrt{21}}{10}.$$

[Maximum mark: 5]

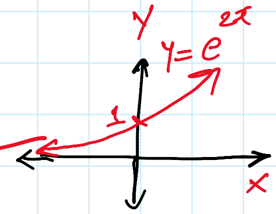
(a) Differentiate the function $f(x) = \dfrac{1-2e^{2x}}{1+e^{2x}}$. *[4 marks]*

(b) Determine whether f is increasing or decreasing. *[1 mark]*

(a) $f'(x) = \dfrac{-6e^{2x}}{(1+e^{2x})^2}$

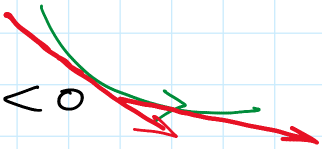
$e^{2x} \neq 0$

No stationary point.



(b) $e^{2x} > 0$

$f'(x) = \dfrac{-6e^{2x}}{(1+e^{2x})^2} < 0$



HW-1

[Maximum mark: 6]

Consider the function $g(x)$ such that $g''(x)$ exists. Let $f(x) = g(e^{2x})$.

(a) Find $f''(x)$ *[5 marks]*

(b) Given that $g'(1) = g''(1) - 1 = 2$, find $f''(0)$ *[1 mark]*

HW-2

Consider the graph G of the function $y = \ln x$ and its reflection G' about the vertical line $x = 3$.

(a) Sketch the graphs of G and G'; indicate the coordinates of the intersection point of the two graphs and the x intercepts. *[2 marks]*

A rectangle ABCD is drawn so that its lower vertices A(p,0) and D are on the x-axis and its upper vertices B and C are on the curves G and G' respectively. The area of this rectangle is denoted by S.

(b) Show that $S = 6\ln p - 2p\ln p$. *[3 marks]*

(c) Given that the maximum value for the area S is obtained at $p=a$ show that

$$a + a\ln a = 3$$ *[3 marks]*

(d) **Hence** find the maximum area of the rectangle. *[2 marks]*

(e) Write down the values of p for which the area S is minimum. *[2 marks]*

HW-3

It is given that x and y satisfy the equation

$$y^4 - \ln\left(\frac{y^2}{4}\right) = x^4 - 6x^2, \quad y > 0.$$

(i) Show that $\dfrac{dy}{dx} = \dfrac{2xy(x^2-3)}{2y^4-1}$. [3]

(ii) Hence obtain the possible exact value(s) of $\dfrac{dy}{dx}$ when $y = 2$. [3]

Q If $y = x + \tan x$, show that

$$\boxed{\cos^2 x \left(\frac{d^2y}{dx^2}\right) - 2y + 2x = 0}$$

Formula:

$$\frac{d}{dx}(\sin x) = \cos x \quad ; \quad \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cos x) = -\sin x \quad ; \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \quad ; \quad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(\text{co}-) = -ve$$

$$y = x + \tan x$$

$$\Rightarrow \frac{dy}{dx} = 1 + \sec^2 x = 1 + (\sec x)^2$$

$$\left\{ \begin{array}{l} y = [f(x)]^n \\ \frac{dy}{dx} = n\,[f(x)]^{n-1} \cdot f'(x) \end{array} \right.$$

$$\Rightarrow \frac{d^2y}{dx^2} = 0 + 2\sec x \cdot (\sec x \tan x)$$

$$= 2\sec^2 x \tan x$$

$$LHS = \cos^2 x \left(\frac{d^2y}{dx^2}\right) - 2y + 2x$$

$$= \cos^2 x \left(2\sec^2 x \tan x\right) - 2(x + \tan x) + 2x$$

$$= 2\cos^2 x \sec^2 x \tan x - 2x - 2\tan x + 2x$$

$$= \cancel{2(1)\tan x} - \cancel{2x} - \cancel{2\tan x} + \cancel{2x}$$

$$= 0$$

$$= RHS$$

$$\left\{ \begin{array}{l} \cos x = \frac{1}{\sec x} \\ \Rightarrow (\cos x \sec x)^2 = 1^2 \\ \Rightarrow \cos^2 x \sec^2 x = 1 \end{array} \right.$$

Chain Rule

Q $y = \sin(\sin x)$

$\sin(f(x))$

Q $y = \sin(\sin x)$ $\qquad \sin(f(x))$

$y' = \cos(\sin x) \cdot \cos x.$

Q $y = e^{\sin x}.$

$y' = e^{\sin x} \cdot \cos x$

$y = e^{\sin(\sin x)}$

$y' = e^{\sin(\sin x)} \cdot \cos(\sin x) \cdot \cos x$

Q $y = \ln(\sin(\sin x))$

$y' = \dfrac{1}{\sin(\sin x)} \cdot \cos(\sin x) \cdot \cos x.$

$uv' + vu'$

Q $y = \ln\left[\sin\left(e^{\cos x} \cdot \sin x\right)\right]$

$y' = \dfrac{1}{\sin(e^{\cos x} \cdot \sin x)} \cdot \cos(e^{\cos x} \cdot \sin x) \cdot \left[e^{\cos x} \cos x + \sin x\, e^{\cos x} \cdot (-\sin x)\right]$

$\qquad = \left((\cos x) e^{\cos x} - \sin^2 x\, e^{\cos x}\right)$

$=$ $\qquad //$

Q $y = e^x$

$y' = e^x$

$y'' = e^x$

$y''' = e^x$

$y^{IV} = e^x.$

$y^n = e^x$

$y = e^{2x}.$

$y' = 2^1 e^{2x}$

$y'' = 2^2 \cdot e^{2x}.$

$y''' = 2^3 \cdot e^{2x}$

$y^{n} = 2^n \cdot e^{2x}$ ✓

HW-4

Find the second order derivatives of the function.

Find the second order derivatives of the function.

$e^x \sin 5x$

HW-5

If $y = x^3 \log x$, prove that $\boxed{\dfrac{d^4y}{dx^4} = \dfrac{6}{x}}$

HW-6

If $y = \sin(\sin x)$, prove that $\dfrac{d^2y}{dx^2} + \tan x . \dfrac{dy}{dx} + y\cos^2 x = 0$

HW-7.

If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $\boxed{x^2 y_2 + xy_1 + y = 0}$

HW-8

If $x = \sin\left(\dfrac{1}{a}\log y\right)$, show that $(1 - x^2)y_2 - xy_1 - a^2y = 0$

HW-9

Find the nth derivative of the function $y = \ln(ax + b)$.

$\cos x \qquad \cos(f(x))$

$-\sin x \qquad -\sin(f(x))\, f'(x)$

$$y' = -\frac{3}{x}\sin(\log x) + \frac{4}{x}\cos(\log x)$$

$$xy' = -3\sin(\log x) + 4\cos(\log x)$$

$$xy_2 + y_1(1) = -\frac{3\cos(\log x)}{x} - \frac{4}{x}\sin(\log x)$$

$$x^2y_2 + xy_1 = -\left[3\cos(\log x) + 4\sin(\log x)\right] = -y$$

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

Q-5 $\qquad y = x^3 \log x$

$\ln x$

$= \log_e x$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$\log_5 x$

$$\cancel{\frac{d}{dx}(\log_5 x) = \frac{1}{x}}$$

$$\frac{\log x}{\log 5}$$

$\log_5 x$

$= \dfrac{\boxed{\log x}}{\log 5}$

$\ln x \to \log_e x$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}(\log_5 x) = \left(\frac{1}{\log 5}\right)\cdot\frac{1}{x}}$$

$$\frac{d}{dx}(\log_5 x) = \frac{d}{dx}\left(\frac{\log x}{\log 5}\right) = \frac{1}{\log 5} \times \frac{1}{x}$$

$$(\log_7 x)' = \frac{1}{\log 7}\cdot\frac{1}{x}$$

Q $\qquad y = \underbrace{x^3 \log_e x}$

$y' = uv' + vu'$

$$\frac{dy}{dx} = x^3 \cdot \frac{1}{x} + (\log x)\, 3x^2$$

$$\boxed{\frac{d^4y}{dx^4} = \frac{6}{x}}$$

$2 \qquad 3x^2 \log x$

$$\frac{dy}{dx} = x^2 \cdot \frac{3}{x} \cdot \ldots$$

$$= x^2 + 3x^2 \log x.$$

$$\boxed{\frac{d^4y}{dx^4} = \frac{6}{x}.}$$

$$\Rightarrow \frac{dy}{dx} = x^2(1 + 3\log x)$$

$$\frac{d^2y}{dx^2} = x^2\left(\frac{3}{x}\right) + (1+3\log x)\,2x$$

$$d \quad \frac{d^2y}{dx^2} = 2x(1+3\log x) + 3x$$

$$\frac{d^3y}{dx^3} = 2(1+3\log x) + 2x\left(\frac{3}{x}\right) + 3$$

$$d \quad \frac{d^3y}{dx^3} = 2(1+3\log x) + 9$$

$$d \quad \frac{d^4y}{dx^4} = \frac{6}{x}$$

Q $\quad y = \ln(ax+b)$, $\quad y^n = ?$

$$y' = \frac{1}{(ax+b)^1} \cdot a$$

$$y'' = \frac{-a^2}{(ax+b)^2}$$

$$y'' = \left(\frac{1}{(ax+b)^2}\right)(-a^2)$$

$$y'' = -a^2\left[(ax+b)^{-2}\right]$$

$$y''' = \frac{-1 \cdot (-2)\; a^3}{(ax+b)^3}$$

$$y''' = -a^2\left[-2(ax+b)^{-3} \cdot a\right]$$

$$= \frac{-1 \cdot -2 \; a^3}{(ax+b)^3}$$

$$y^{IV} = -1 \cdot -2 \cdot -3 \; \frac{a^4}{(ax+b)^4}$$

$$y^n = (-1)^{n-1}(n-1)!\,\frac{a^n}{(ax+b)^n}$$

$$\begin{cases} 1 & 0! \\ 1 & 1! \\ 1 \times 2 & 2! \\ 1 \times 2 \times 3 & 3! \\ 1 \times 2 \times 3 \times 4 = 4! & \end{cases}$$