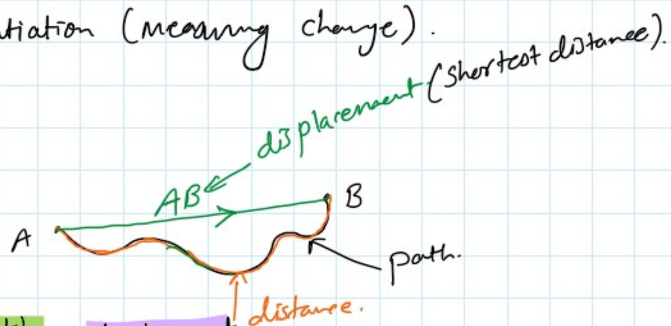
Kinematics and derivative of exponential function

Thursday, January 21, 2021 5:48 AM

Study of movement of objects

change

Differentiation (meaning change).



$y = S(t)$ displacement

Rate of change. (Differentiation).

$v(t) = s'(t) = \frac{ds}{dt}$ ← velocity

$a(t) = \frac{d^2s}{dt^2} = s''(t) = v'(t)$ ← acceleration.

Right.

$\left.\begin{matrix} v(t) \\ a(t) \end{matrix}\right\} \begin{matrix} + \\ + \end{matrix}$ speeding up. (accelerating)

$\left.\begin{matrix} v(t) \\ a(t) \end{matrix}\right\} \begin{matrix} - \\ - \end{matrix}$ speeding up.

* v & a have the same sign, an object is speeding up (accelerating).
* v & a have different signs, an object is slowing down (decelerating)

A particle moves in a horizontal line so that its position from a fixed point after t seconds, where $t \geq 0$, is s metres, where $s(t) = 5t^2 - t^4$.

a Find the position, velocity and acceleration of the particle after 1 second.

b Determine whether the particle is speeding up or slowing down at $t = 1$.

c Find the values of t when the particle is at rest.

d Find the time intervals on which the particle is speeding up, and the intervals on which it is slowing down.

e Find the total distance the particle travels in the first 3 seconds.

$s(t) = 5t^2 - t^4$ — position of object at t

Differentiation w.r.t time (rate of change).

$v(t) = \frac{ds}{dt} = 10t - 4t^3$

$a = \frac{v}{t} = \frac{m/s}{s}$

$\frac{dv}{dt} = s''(t) = 10 - 12t^2$

$V(t) = \frac{ds}{dt} = 10t - 4t^3$

$a(t) = \frac{dv}{dt} = s''(t) = 10 - 12t^2$

$a = \frac{v}{t} = \frac{m/s}{s}$

$= m/s^2$

$t = 1.$

$s(1) = 5(1)^2 - 1^4 = 5 - 1 = 4 \text{ m}$

$V(1) = 10(1) - 4(1)^3 = 10 - 4 = 6 \text{ m/s}$

$a(1) = 10 - 12(1)^2 = 10 - 12 = -2 \text{ m/s}^2$

b) Since v and a have opposite signs at $t=1$ the particle is slowing down.

c) Rest.

$V(t) = 0$

$10t - 4t^3 = 0$

$2t(5 - 2t^2) = 0$

$t = 0$, $t = \pm\sqrt{\frac{5}{2}}$

$t = \sqrt{\frac{5}{2}}$

t	$t < \sqrt{5/2}$	$t > \sqrt{\frac{5}{2}}$
sign $V(t)$	+	−

$a^2 - b^2 = (a+b).(a-b)$

d) $a(t) = 0$

$10 - 12t^2 = 0$

$\Rightarrow t = \sqrt{\frac{5}{6}}$

$\boxed{t \geq 0}$

$\frac{5}{6} - t^2 = 0$

$\left(\sqrt{\frac{5}{6}}\right)^2 - t^2 = 0 \Rightarrow \left(\frac{\sqrt{5}}{\sqrt{6}} - t\right)\left(\frac{\sqrt{5}}{\sqrt{6}} + t\right) = 0$

t	$0 \leq t < \sqrt{\frac{5}{6}}$	$\frac{\sqrt{5}}{\sqrt{6}} < t < \sqrt{\frac{5}{2}}$	$t > \sqrt{5/2}$
sign $V(t)$	+	+	−
sign $a(t)$	+	−	−

The particle is speeding up when

$0 < t < \sqrt{\frac{5}{6}}$ & $t > \sqrt{\frac{5}{2}}$

slowing down in $\sqrt{\frac{5}{6}} < t < \sqrt{\frac{5}{2}}$

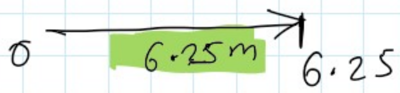
(e) $t = 0$, $s(t) = 5t^2 - t^4$

$s(0)=0.$

$t=\sqrt{\frac{5}{2}}$

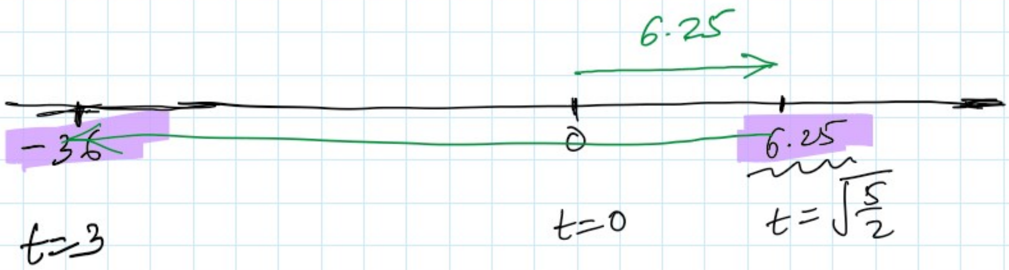
$$s\left(\sqrt{\frac{5}{2}}\right) = 5\left(\frac{\sqrt{5}}{\sqrt{2}}\right)^2 - \left(\frac{\sqrt{5}}{\sqrt{2}}\right)^4$$

$$s\left(\frac{\sqrt{5}}{\sqrt{2}}\right) = 5\times\frac{5}{2} - \frac{25}{4} = \frac{25}{4} = 6.25\text{ m.}$$



$\boxed{t=3}$

$$s(3) = 5(3)^2 - 3^4 = -36.$$

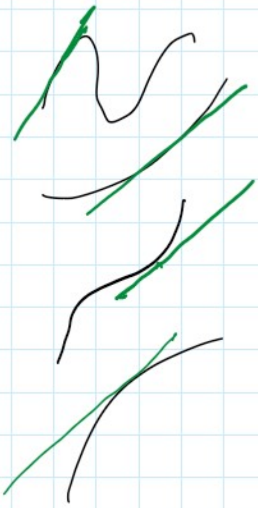


$t=3 \Rightarrow 6.25-(-36) = 42.25$ m.

Total distance travelled $= 42.25+6.25$
$= 48.5$ m

Differentiation:-

1) Algebraic
2) trigonometric — $y=\sin x$
3) $\boxed{\text{Exponential}}$ — $y=e^x$
4) Inverse. — $y=\sin^{-1}x$
5) logarithmic. — $y=\log x.$



Exponential equation:-

a) $3^{x+2} = 27$

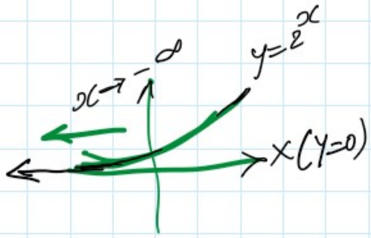
$x = 1$

b) $4^{2x+1} = 8^{x+2}$

$x = 4$

Consider the function $y = 2^x + 3$.

- **a** Find **i** the y-intercept **ii** the equation of the asymptote.
- **b** State the domain and range of the function.
- **c** Sketch the graph of the function, showing the asymptote as a dotted line.
- **d** Using your GDC, solve the equation $2^x + 3 = 2 + 3^{-x}$.



a) $x = 0$, $y = 2^0 + 3 = 4$

y-intercept is $(0, 4)$

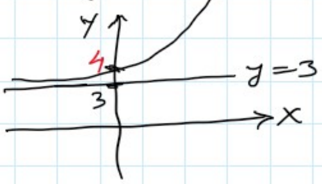
(ii) $y = 3$

$x \to -\infty$, $y \to$

$y = 2^x + 3$

$2^x \to 2^{-\infty} \to 0$

$y \to 0 + 3 = 3$



(d) $2^x + 3 = 2 + 3^{-x}$

$x = -0.490$

* Natural log.

* Diff. of exp. & log...

* Euler (e).
Value.

The value of a boat, y, in thousands of UK pounds (£) is modelled by the function $y = 20(0.85)^x$, where x is the number of years since the boat was manufactured.

- **a** Find the value of the boat when it was brand new.
- **b** Estimate the value of the boat when it is 3 years old. Give your answer to the nearest pound.
- **c** Use your GDC to estimate when the value of the boat will be worth half its original value.

1 Solve each equation:

- **a** $2^x = 16$ **b** $10^x = 1\,000\,000$
- **c** $2^{x+1} = 64$ **d** $3^{2x-1} = 27$
- **e** $3^{1-2x} = 1$ **f** $3 \times 2^x = 48$
- **g** $4^{x+2} = \frac{1}{64}$ **h** $\sqrt[4]{3} = 9^x$
- **i** $\left(\frac{1}{5}\right)^x = 25$ **j** $2^x = 2\sqrt{2}$

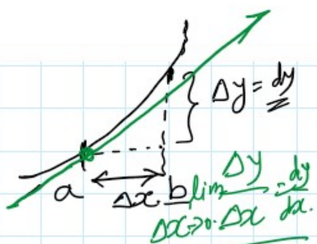
2 Convert to the same base and solve each equation:

- **a** $2^{x+3} = 4^{x-2}$ **b** $5^{x-3} = 25^{x-4}$
- **c** $6^{2x-6} = 36^{3x-5}$ **d** $9^{5x+2} = \left(\frac{1}{3}\right)^{11-x}$

$f(x) = e^x$, $f'(x) = ?$

$f(x+h) = e^{x+h}$, $x \to x+h$.

$\frac{dy}{dx} = \lim \frac{\text{change in } y}{\text{change in } x}$



$\Delta x \to 0$

first principle.

$\frac{d}{dx}=$ change in x

$\Delta x \to 0$

first principle.

$$\frac{dy}{dx} = \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\right\}$$

$$= \lim_{h\to 0}\left[\frac{(e^{x+h}-e^{x})}{h}\right]$$

$$= \lim_{h\to 0} \frac{e^{x}\cdot e^{h}-e^{x}}{h}$$

$$= \lim_{h\to 0} \frac{e^{x}(e^{h}-1)}{h}$$

$$= e^{x}\lim_{h\to 0}\left(\frac{e^{h}-1}{h}\right)$$

$$= e^{x}(1)$$

$$\boxed{\frac{dy}{dx}=e^{x}}$$

① $f(x)=e^{x}$, $f'(x)=e^{x}$.

② $f(x)=\ln x$, $f'(x)=\frac{1}{x}$.

Ex: find derivative

ⓐ $h(x)=e^{x}\cdot \ln x$.

$u=e^{x}$
$v=\ln x$.

$h(x)=u\cdot v$

$h'(x)=u\cdot v'+v\cdot u'$

$$h'(x)=e^{x}\cdot\frac{1}{x}+\ln x\cdot(e^{x})$$

$$=e^{x}\left(\frac{1}{x}+\ln x\right)$$

ⓑ $f(x)=\frac{e^{x}}{\ln x}=\frac{u}{v}$

$$f'(x)=\frac{vu'-uv'}{v^{2}}$$

$$f'(x)=\frac{(\ln x)e^{x}-e^{x}\cdot\frac{1}{x}}{(\ln x)^{2}}$$

$$=\frac{e^{x}\left(\ln x-\frac{1}{x}\right)}{(\ln x)^{2}}$$

Ex $g(x)=e^{2x}$.

$u=2x$. $\leftarrow$ $\frac{du}{dx}=2$

$g(x)=e^{u}$

$g'(x)=e^{u}\cdot\frac{du}{dx}$

$y=(2x+3)^{3}$

$y'=3(2x+3)^{2}\cdot(2+0)$

chain rule.

$y'=6(2x+3)^{2}$

$$\begin{cases} g(x) = e^{u} \\ g'(x) = e^{u} \cdot \dfrac{du}{dx} \\ \quad = e^{2x} \cdot 2 \end{cases}$$

Chain Rule.

$$y' = 6(2x+3)^2$$

$$\begin{cases} g(x) = e^{x^2} \\ g'(x) = e^{x^2} \cdot 2x \end{cases}$$

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$$

Ex

$$y = \ln(3x^5)$$

$$\frac{dy}{dx} = \frac{1}{3x^5} \times \frac{d}{dx}(3x^5)$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$y = u \cdot v$$

Ex

$$y = (5x^2+3)^{1/3} \qquad x^n$$

$$\frac{dy}{dx} = \frac{1}{3}(5x^2+3)^{\frac{1}{3}-1} \cdot (10x)$$

$$= \frac{10x}{3}(5x^2+3)^{-2/3}$$

Ex

$$y = \ln(3x^5) \qquad \ln x$$

$$\frac{dy}{dx} = \frac{1}{3x^5} \cdot 15x^4$$

$$y \;-\; u \;-\; x$$

$$= \frac{dy}{du} \times \frac{du}{dx}$$

$$y = \ln(3x^5)$$

$$u = 3x^5$$

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$$

$$y = \ln u$$

$$\frac{dy}{du} = \frac{1}{u}, \qquad u = 3x^5, \quad \frac{du}{dx} = 15x^4$$

$$= \frac{1}{u} \cdot 15x^4$$

$$= \frac{1}{3x^5} \cdot 15x^4$$

Ex

$$y = \ln(e^x \cdot x^3)$$

Chain Rule.

$$\frac{dy}{dx} = \frac{1}{e^x \cdot x^3} \times \frac{d}{dx}(e^x \cdot x^3)$$

$$= \frac{1}{e^x \cdot x^3} \cdot \left[e^x \cdot 3x^2 + x^3 \cdot e^x \right]$$

$$= \frac{\cancel{e^x} \cdot \cancel{x^2}(3+x)}{\cancel{e^x} \cdot \cancel{x^3}}$$

$$= \left(\frac{3+x}{x}\right)$$

$uv' + vu'$

$vu' + uv'$

Ex $y = 2(x^3 \cdot e^{-3x})$

$$\frac{dy}{dx} = 2\left[3x^2 e^{-3x} + x^3 e^{-3x} \cdot (-3)\right]$$

Ex $y = x^2 \ln(2x+3)$

$$\frac{dy}{dx} =$$

Ex $y = e^{(4x^3+5)^2}$

$$\frac{dy}{dx} = e^{(4x^3+5)^2} \cdot \frac{d}{dx}(4x^3+5)^2$$

$$= e^{(4x^3+5)^2} \cdot \left[2(4x^3+5) \cdot 12x^2\right]$$

Ex $y = \ln\left(\frac{x^2+1}{x^3-x}\right)$

$y = \ln x$

$\frac{dy}{dx} = \frac{1}{x}$

$$y' = \frac{x^3-x}{x^2+1} \cdot \left(\frac{d}{dx}\left(\frac{x^2+1}{x^3-x}\right)\right)$$

$$y = \ln\left(\frac{x^2+1}{x^3-x}\right)$$

$\ln\left(\frac{m}{n}\right) = \ln m - \ln n$

$$y = \ln(x^2+1) - \ln(x^3-x)$$

$\ln(m \times n) = \ln m + \ln n$

$$\frac{dy}{dx} = \frac{2x}{x^2+1} - \frac{3x^2-1}{x^3-x}$$

Ex $y = \frac{2e^{4x}}{1-e^x}$

Ex

$$y = \frac{2e^{4x}}{1-e^{x}}.$$

$$\frac{dy}{dx} = \frac{(1-e^{x})\,8\cdot e^{4x} - 2e^{4x}(-e^{x})}{(1-e^{x})^2}$$

$$= \frac{2e^{4x}(4-4e^{x}+e^{x})}{(1-e^{x})^2}$$

$$= \frac{2e^{4x}(4-3e^{x})}{(1-e^{x})^2}$$

Ex R

$$y = \frac{e^{x}+1}{e^{x}-1} \quad \left(\frac{u}{v}\right)$$

$$\boxed{y = \frac{u}{v} \qquad \frac{dy}{dx} = \frac{vu' - uv'}{v^2}}$$

$$\frac{dy}{dx} = \frac{(e^{x}-1)(e^{x}) - (e^{x}+1)e^{x}}{(e^{x}-1)^2}$$

$$= \frac{e^{x}[\cancel{e^{x}} - 1 - \cancel{e^{x}} - 1]}{(e^{x}-1)^2}$$

$$= \frac{-2e^{x}}{(e^{x}-1)^2}$$

Ex R

$$y = \frac{2-\ln x}{x}.$$

$$\frac{dy}{dx} = \frac{\cancel{x}(-1/\cancel{x}) - (2-\ln x)\cdot 1}{x^2}$$

$$= \frac{-1 - 2 + \ln x}{x^2}$$

$$= \frac{-3+\ln x}{x^2}$$