Worksheet

Tuesday, February 23, 2021 8:18 AM

1. Find the local minima and the local maxima of each given function.

   (a) $f(x, y) = \sin x \sin y$

   (b) $g(x, y) = 2 - x^4 + 2x^2 - y^2$

   (c) $h(x, y) = x \sin y$

2. Find the absolute(global) minimum and maximum values of $f(x, y) = xy^2$ on $\Omega = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$.

3. Use the second derivative test to find the three positive numbers whose sum is 90 and whose product is maximum.

4. Find the shortest distance from the origin to the plane $x + y + z = 2$.

1. Find the extreme values of $f(x, y, z) = z$, subject to constraints $x^2 + y^2 = z^2$, and $x + y + z = 10$.

2. Find each given integral

   (a) $\int_D \int (x + 1) dA, \ D = \{(x, y) : |x| \leq 1, 0 \leq y \leq 2\}$

   (b) $\int_0^1 \int_0^1 (2x - y)^2 dx dy$,

   (c) $\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (1 + x^2 \sin y + y^2 \sin x) dx dy$

   (d) $\int_D \int (x + y) dA, \ D = \{(x, y) : 1 \leq x \leq 2, y - 1 \leq x \leq 1\}$

$\left\{ \begin{matrix} -\pi \leq x \leq \pi \\ -\pi \leq y \leq \pi \end{matrix} \right\}$

1) $f(x,y) = \sin x \sin y$

$f_x(x,y) = \cos x \sin y$ & $f_y(x,y) = \sin x \cos y$

$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 0, 0 \rangle$

$f_x(x,y) = 0 \Rightarrow \cos x \sin y = 0$

$\cos x = 0$ or $\sin y = 0$

$x = \pm(2n+1)\frac{\pi}{2}$ $\quad y = n\pi$

$n \in \{0, 1, 2, \dots\}$

$f_y(x,y) = 0 \Rightarrow \sin x \cos y = 0$.

$* \quad x = \pm(2n+1)\frac{\pi}{2} \longrightarrow \sin\left[\pm(2n+1)\frac{\pi}{2}\right] \cos y = 0$.

$\Rightarrow \cos y = 0$

$y = \pm (2n+1)\frac{\pi}{2}$

$** \quad y = n\pi \longrightarrow \sin x \cos(n\pi) = 0$

$\Rightarrow \sin x = 0$

$x = n\pi$

the critical points are. $\quad D(x,y) \quad f_{xx}$

① $\left((2n+1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\right)$ $\quad D > 0 \quad f_{xx} < 0$ — local Max

② $\left((2n+1)\frac{\pi}{2}, -(2n+1)\frac{\pi}{2}\right)$ $\quad D > 0 \quad f_{xx} > 0$ — Local Min

③ $\left(-(2n+1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\right)$ $\quad D > 0 \quad f_{xx} > 0$ — local Min

④ $\left(-(2n+1)\frac{\pi}{2}, -(2n+1)\frac{\pi}{2}\right)$ $\quad D > 0 \quad f_{xx} < 0$ — local max.

saddle point

$f_{xx} < 0$ — loc.

④ $\left(-(2n+1)\frac{\pi}{2}, -(2n+1)\frac{\pi}{2} \right)$ ✓ $D > 0$

⑤ $(n\pi, n\pi)$ $D < 0$ —— saddle point

$$D(x,y) = f_{xx} \cdot f_{yy} - [f_{xy}]^2$$

$\because$ $f_{xx} = -\sin x \sin y$ & $f_{yy} = -\sin x \sin y$

& $f_{xy} = \cos x \cos y$

$$D(x,y) = (-\sin x \sin y)(-\sin x \sin y) - (\cos x \cos y)^2$$

$$D(x,y) = \sin^2 x \sin^2 y - (\underline{\cos x \cos y})^2$$

$$D\left[(2n+1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\right] = 1 \times 1 - [0 \times 0]^2 = 1 > 0$$

$$f_{xx}\left[(2n+1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\right] = -\sin\left[(2n+1)\frac{\pi}{2}\right]\sin\left[(2n+1)\frac{\pi}{2}\right]$$
$$= -1 < 0.$$

$\therefore$ $\left[(2n+1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\right]$ will local max.

②
$$g(x,y) = 2 - x^4 + 2x^2 - y^2$$

$$g_x(x,y) = -4x^3 + 4x \quad \Big| \quad g_y(x,y) = -2y$$

$g_x(x,y) = 0$	$g_y(x,y) = 0$
$-4x^3 + 4x = 0$	$-2y = 0$
$-4x(x^2 - 1) = 0$	$y = 0.$
$-4x(x+1)(x-1) = 0$	
$\underline{x = 0}$ or $x = -1, x = 1$	

Critical points are:

$(0,0)$

$(-1,0)$

$(1,0)$

$$g_{xx}(x,y) = -12x^2 + 4$$

$$g_{yy}(x,y) = -2$$

$$g_{xy}(x,y) = 0$$

$$D(x,y) = (-12x^2 + 4)(-2) - 0^2$$

$$D = 24x^2 - 8$$

	$D(x,y)$	$g_{xx}(x,y)$	
$(0,0)$	$-8 < 0$		← saddle point.
$(-1,0)$	$16 > 0$	$-8 < 0$	local Max.
$(1,0)$	$16 > 0.$	$-8 < 0$	local Max.

③ $h(x,y) = x\sin y$

$h_x(x,y) = \sin y$ & $h_y(x,y) = x\cos y$

$h_x(x,y) = 0$

$\sin y = 0$

$y = \underline{0}, \ \underline{\pm n\pi}$

$h_y(x,y) = 0$

$x\cos y = 0$

$h_{xx} = 0$

$h_{yy} = -x\sin y$

$h_{xy} = \cos y$

$\xrightarrow{y=0}$ $x\cos(0) = 0$

$\underline{x = 0}$.

$\xrightarrow{y=\pm n\pi}$ $x\cos(\pm n\pi) = 0$

$\underline{x = 0}$.

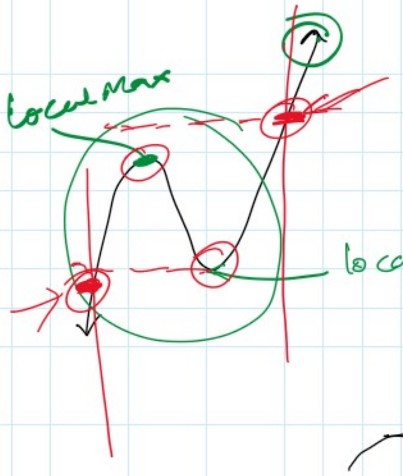
Critical points are.

$\begin{cases} (0,0) \\ (0, \pm n\pi) \end{cases}$

$D = 0(-x\sin y) - \cos^2 y$

$= -\cos^2 y$

$\left.\begin{array}{l} D < 0 \\ D < 0 \end{array}\right\}$ — saddle point



global max, ∞

global Min $-\infty$

② $f(x,y) = xy^2$, $\Omega = \{(x,y) : x \geq 0, y \geq 0, \ x^2+y^2 \leq 1\}$

$\begin{cases} \text{Critical points} \\ \text{boundary points.} \end{cases}$

$f_x(x,y) = y^2$, $f_y(x,y) = 2xy$

$f_x(x,y) = 0$

$y^2 = 0$

$\Rightarrow y = 0$

$f_y(x,y) = 0$

$2xy = 0$

$x = 0$ or $y = 0$.

(1) $(0,0)$ ←

Boundary condition. $x^2+y^2=1$

$y^2 = 1-x^2$ — (1)

$\because f(x,y) = xy^2$ — (2)

pluging (1) in (2) to make the function as single variable f^n.

$g(x) = x(1-x^2)$

$g(x) = x - x^3$

$g'(x) = 1-3x^2$

$g'(x)=0$

$1-3x^2=0$

$\Rightarrow 3x^2=1$

$\Rightarrow x = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}}$

$\because y^2 = 1-x^2 \Rightarrow y^2 = 1-\frac{1}{3} = \frac{2}{3}$

$y = \pm\sqrt{\frac{2}{3}}$

$f(x,y) = xy^2$

$(0,0)$	0	
✓ $\left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$	$\frac{1}{\sqrt{3}} \times \frac{2}{3} = \frac{2}{3\sqrt{3}}$	global max
✓ $\left(\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right)$	$\frac{1}{\sqrt{3}} \times \frac{2}{3} = \frac{2}{3\sqrt{3}}$	
✓ $\left(-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$	$-\frac{1}{\sqrt{3}} \times \frac{2}{3} = \frac{-2}{3\sqrt{3}}$	global min.
✓ $\left(-\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{3}}\right)$	$-\frac{1}{\sqrt{3}} \times \frac{2}{3} = \frac{-2}{3\sqrt{3}}$	

global max appears at $\left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right), \left(\frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}\right)$

global min ———— $(\quad) \cdot (\quad)$

(3) $x+y+z=90$,

Maximise(xyz)

$z = 90 - x - y$

Maximize $f(x,y) = xy(90-x-y)$

$= 90xy - x^2y - xy^2$

$f_x(x,y) = 90y - 2xy - y^2$, $f_y(x,y) = 90x - x^2 - 2xy$

$f_x(x,y) = 0$

$90y - 2xy - y^2 = 0$

$y(90 - 2x - y) = 0$

$y = 0$ or $90 - 2x - y = 0$

$y = 90 - 2x.$

for $y = 0$, $90x - x^2 - 2x(0) = 0$

$90x - x^2 = 0$

$\Rightarrow x = 0$ or $x = 90.$

C.P. $(0,0), (90,0)$.

for $y = 90 - 2x.$, $90x - x^2 - 2x(90-2x) = 0$

$90x - x^2 - 180x + 4x^2 = 0$

$3x^2 - 90x = 0$

$x = 0$ or $x = \frac{90}{3} = 30.$

$y = 90$, $y = 90 - 2(30)$

$y = 30$

C.P. $(0,90)$, $(30,30)$

$f_{xx} = -2y$

$f_{yy} = -2x.$

$f_{xy} = 90 - 2x - 2y$

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$

$D(x,y) = -2y(-2x) - [90 - 2x - 2y]^2$

$(0,0)$; $D = 0 - (90)^2 = -8100 < 0$
$(0,90)$; $D = -[90-180]^2 < 0$
$(90,0)$; $D = < 0.$

Saddle points.

$(30,30)$; $D = -2(30)(-2\times 30) - [90 - 60 - 60]^2$
$= 3600 - 900 > 0$

$f_{xx}(30,30) = -2(30) = -60 < 0.$

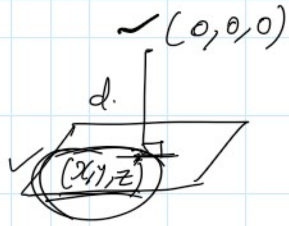
$(30,30)$ is local Max.

$\therefore$ f is maximize at $(30,30)$

$z = 90 - 30 - 30 = 30$

$(30,30,30)$

(4) Lagrange's multiplies
$x+y+z=2$



$d = \sqrt{x^2+y^2+z^2}$

Subject to $x+y+z=2$

Minimize $d^2 = x^2+y^2+z^2$ subject to $x+y+z=2$
$f(x,y,z) = x^2+y^2+z^2$ $\quad g(x,y,z) = x+y+z-2=0$

$\nabla f = \lambda \nabla g$ — Lagrange's multiplier.

$\langle f_x, f_y, f_z \rangle = \lambda \langle g_x, g_y, g_z \rangle$

$\langle 2x, 2y, 2z \rangle = \lambda \langle 1, 1, 1 \rangle$

$2x = \lambda \Rightarrow \boxed{x = \frac{\lambda}{2}}$
$2y = \lambda \Rightarrow y = \frac{\lambda}{2}$
$2z = \lambda \Rightarrow z = \frac{\lambda}{2}$

$x+y+z = 2$

$\frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{2} = 2$

$3\lambda = 4$

$\lambda = 4/3.$

$x = \frac{4/3}{2} = \frac{4}{6.} = \frac{2}{3}$

$y = \frac{4/3}{2} = 4/6 = 2/3$

$z = \frac{2}{3}$

$d = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{4}{9}}$

$= \sqrt{\frac{12}{9}} = \sqrt{\frac{4}{3}} = \underline{\underline{\frac{2}{\sqrt{3}}}}$

$= \frac{2\sqrt{3}}{3}$

1) $f(x,y,z) = \underline{\underline{z}}$ subject to $x^2+y^2 = z^2$

$x+y+z = 10.$

$h(x,y,z) = x^2+y^2-z^2 = 0$

$\nabla f = \lambda \nabla g + \mu \Delta h$

$g(x,y,z) = \underline{x+y+z-10 = 0}$

$\langle 0,0,1\rangle = \lambda \langle 1,1,1\rangle + \mu \langle 2x, 2y, -2z\rangle$

$0 = \lambda + 2\mu x, \quad 0 = \lambda + 2\mu y \quad \& \quad 1 = \lambda - 2\mu z$

$\Rightarrow \lambda = -2\mu x, \quad \lambda = -2\mu y$

$-2\mu x = -2\mu y$

$\underline{\mu \neq 0} \quad \boxed{x = y} \quad — (1)$

if $\cancel{\mu = 0}$ $\quad \langle 0,0,1\rangle = \lambda\langle 1,1,1\rangle$

$\boxed{0 = \lambda} \quad \boxed{1 = \lambda}$

plugging (1) in $g(x,y,z)$

$x + x + z - 10 = 0$

$2x + z - 10 = 0$

$2x = 10 - z$

$\Rightarrow x = 5 - \frac{z}{2} \quad — (2) \qquad y = 5 - \frac{z}{2}$

plugging (1) & (2) in second constrained $h(x,y,z) = x^2+y^2-z^2 = 0$

$\left(5 - \frac{z}{2}\right)^2 + \left(5 - \frac{z}{2}\right)^2 - z^2 = 0$

$$\left(5-\frac{z}{2}\right)^2+\left(5-\frac{z}{2}\right)^2-z^2=0$$

$$\Rightarrow 2\left(5-\frac{z}{2}\right)^2-z^2=0$$

$$\Rightarrow 2\left[25-5z+\frac{z^2}{4}\right]-z^2=0$$

$$\Rightarrow 50-10z+\frac{z^2}{2}-z^2=0$$

$$\Rightarrow 100-20z+z^2-2z^2=0$$

$$\Rightarrow 100-20z-z^2=0.$$

$$\Rightarrow z^2+20z-100=0.$$

$$z=\frac{-20\pm\sqrt{400-4(-100)}}{2}$$

$$z=\frac{-20\pm 20\sqrt{2}}{2}=-10\pm 10\sqrt{2}$$

$$=-10+10\sqrt{2},\ -10-10\sqrt{2}$$

$f(x,y,z)=z$

$f(x,y,z)=-10+10\sqrt{2}>0 \leftarrow$ max.

$f(x,y,z)=-10-10\sqrt{2}<0 \leftarrow$ min.