FINITE in Infinite

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Series & convergence.

Convergence of infinite series.

$\lim_{n \to \infty} u_n \to L$ — convergent [For sequence]

If $u_n = \left(\frac{1}{n}\right)$ $\lim_{n \to \infty} u_n = 0$ (finite value)

$u_n = \frac{1}{n}$ (convergent).

Definition: If the sequence of partial sum has a Limit L as $n \to \infty$, then the series converges to the limit L

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots = \boxed{\sum_{K=1}^{\infty} u_K = L}$$

When an infinite series has a real number as a sum, we say that it is a convergent series, and it converges to its sum.

Geometric series:

$|r| < 1$

$$1 + r + r^2 + r^3 + \cdots + r^n + \cdots \infty$$

common ratio $= r$

$|r| < 1 : -1 < r < 1$

$$\text{Sum} = \frac{1(1-r^{n+1})}{1-r} \quad (\text{for } n \text{ terms})$$

$\left(\frac{1}{2}\right)$

$n \to \infty$ $\left(\frac{1}{3}\right)$

$(r)^{n+1} \to 0$

$$S_\infty = \frac{1}{1-r}$$ ← First term.

Ex verify that series $2+\frac{3}{2}+\frac{9}{8}+\frac{27}{32}+\cdots$ converges and find its sum.

$\longrightarrow$ decreasing series.

Sol^n

$$r_1 = \frac{3/2}{2} = \frac{3}{4}$$

$$r_2 = \frac{9/8}{3/2} = \frac{3}{4}$$

So, $r = \frac{3}{4} < 1$.

$|r| < 1$

$$S_\infty = \frac{2}{1-\frac{3}{4}} = 8$$

$2+2^2+2^3+2^4+2^5\cdots$

$r=2$ $\not\to L$

1) Divergence Test:-

If $\lim_{k\to\infty} u_k \neq 0$ or if the limit does not exist, the series $\sum_{k=1}^{\infty} u_k$ is divergent.

Ex Show that the series $\sum_{k=1}^{\infty} \frac{k^2+3k+1}{4k^2+3}$ diverges.

Sol^n

$$\lim_{k\to\infty} \frac{k^2+3k+1}{4k^2+3} = \frac{1}{4} \neq 0$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{k^2+3k+1}{4k^2+3} \text{ is diverges.}$$

If $\lim_{k\to\infty} u_k = 0$ } Inconclusive. (may be Diverges or converges)

$u_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = \underline{\underline{0}}$ } I'll prove this to be divergent.

Ex! converges/diverges.

a) $\frac{1}{2} + \frac{3}{2} + \frac{3}{4} + \cdots + \frac{n}{n+1} + \cdots$

$\lim_{n\to\infty} \frac{n}{n+1} = 1. \neq 0$ (Divergent)

b) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ (Harmonic Series)

$\lim_{n\to\infty} \frac{1}{n} = 0$ (Inconclusive).

Harmonic Series is divergent!-

Assume $\sum_{n=1}^{\infty} \frac{1}{n} = \underline{\underline{S}}$. (finite).

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \cdots$$

$$= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \cdots + \left(\frac{1}{n} + \frac{1}{n+1}\right) + \cdots$$

$$> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \cdots + \left(\frac{1}{n+1} + \frac{1}{n+1}\right)$$

$$> \left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{1}{6}\right)+\cdots+\left(\frac{1}{n+1}+\frac{1}{n+1}\right)+\cdots$$

$$S > \underbrace{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots}=S \text{ (finite)}$$

$$S > S$$

an absurd result

Sum of harmonic series is not a finite value.

$\therefore$ Series is divergent.

Introduction to convergence test for series:-

The integral as the limit of sums.

Definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x.$$

$\longrightarrow$ finite. $\longrightarrow$ finite.

If I put ∞ in integral (in place limit a, b)

Improper Integral.

If $f(x)$ is continuous on $[a,\infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

Convergence of an improper integral:-

Convergence of an improper integral:-

Ex

$$\int_{1}^{\infty} \frac{1}{x}\, dx.$$

$$= \lim_{b\to\infty} \int_{1}^{b} \frac{1}{x}\, dx.$$

$$= \lim_{b\to\infty} \left[\ln x\right]_{1}^{b}$$

$$= \lim_{b\to\infty} (\ln b - 0) = \infty$$

the integral is diverges.

b) $$\int_{0}^{\infty} \frac{1}{1+x^2}\, dx.$$

$$= \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\, dx.$$

$$= \lim_{b\to\infty} \left[\arctan x\right]_{0}^{b}$$

$$= \lim_{b\to\infty} \arctan b$$

$$= \boxed{\frac{\pi}{2}}$$

$$\int_{0}^{\infty} \frac{1}{1+x^2}\, dx = \frac{\pi}{2}$$

integral is converges to $\pi/2$

Integral test for convergence:-

Let $f(x)$ be a continuous, positive & decreasing function. Then the series $f(1) + f(2) + \cdots + f(n) + \cdots$

converges if the improper integral

$\int_1^\infty f(x)\,dx$ converges & diverges if the

integral diverges.

Ex: Determine if $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges or diverges.

Soln:

1) $\lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0$ (decreasing sequence)

$\uparrow$ inconclusive.

2) Integral test.

$$\int_1^\infty f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x\sqrt{x}}\,dx.$$

$$= \lim_{b\to\infty} \left[-2x^{-1/2}\right]_1^b$$

$$= \lim_{b\to\infty} \left(-\frac{2}{\sqrt{b}} + 2\right) = \underline{\underline{2}}$$

$$= \lim_{b\to\infty} \left(-\frac{\ }{\sqrt{b}} + 2\right) = 2$$

The series is convergent.

The p-Series test:-

1) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$

2) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 1$

#Ex $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$; $p = \frac{1}{3} < 1$

Diverges.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by integral test.

a) $\lim_{n\to\infty} \frac{1}{n^2} = 0$

b) $\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx.$

$= \lim_{b\to\infty} \left[\frac{-1}{x}\right]_1^b$

$= \lim_{b\to\infty} \left[-\frac{1}{b} + 1\right] = 1.$

Series is converges.

Ex

$$\sum_{n=1}^{\infty}\left(\frac{1}{(2n+1)^5}\right)$$

$p>1$