Optimization

Wednesday, June 24, 2020 4:03 PM

Second derivative Test:- (Hessian Matrix)

$$D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - (f_{xy})^2$$

If we get

1) $D(a,b) > 0$, $f_{xx}(a,b) > 0$ $\Rightarrow$ concave up. (Relative min)
   $f_{yy}(a,b) > 0$

2) $D(a,b) > 0$, $f_{xx}(a,b) < 0$ $\Rightarrow$ concave down (Relative max).
   $f_{yy}(a,b) < 0$

3) $D(a,b) < 0$, $\begin{Bmatrix} f_{xx}(a,b) < 0 \\ f_{yy}(a,b) > 0 \end{Bmatrix}$ or $\begin{Bmatrix} f_{xx}(a,b) > 0 \\ f_{yy}(a,b) < 0 \end{Bmatrix}$ Saddle point.
   (Concave up & down at one point)

4) $D(a,b) = 0$, Inconclusive.

Ex: $f(x,y) = x^2 + 5y^2 + x^2y + 2y^3$

$f_x = 2x + 2yx$, $f_y = 10y + x^2 + 6y^2$

critical points: $(0,0)$, $(0,-\frac{5}{3})$, $(2,-1)$, $(-2,-1)$

$f_{xx} = 2 + 2y$, $f_{yy} = 10 + 12y$; $f_{xy} = 2x$.

$D(x,y) = (2+2y)(10+12y) - 2x$.

✓ $(0,0)$: $D(0,0) = 2 > 0$; $f_{xx}(0,0) = 2 > 0$ Relative Minima
$(0,0,2)$

✓ $(0,-\frac{5}{3})$: $D(0,-\frac{5}{3}) = \frac{40}{3} > 0$; $f_{xx}(0,-\frac{5}{3}) = \frac{-4}{3} < 0$ Relative Maxima.
$(0,-\frac{5}{3},\frac{125}{27})$

$(2,-1)$: $D(2,-1) = -4 < 0 \Rightarrow$ Saddle point.

$Z =$

$(-2,-1)$: $D(-2,-1) = 4 > 0$, $\boxed{f_{xx} = 0}$ — Relative maxima.
$\boxed{f_{yy} = -2}$

$Z =$

Ex. $f(x,y) = e^{-x^2-y^2}$

Soln:

$-x^2-y^2$ $-x^2-y^2$

Soln:

$$f_x = -2x e^{-x^2-y^2}, \quad f_y = -2y\, e^{-x^2-y^2}$$

$$f_x = 0 \qquad , \quad -2y e^{-x^2-y^2} = 0$$

$$\Rightarrow -2x e^{-x^2-y^2} = 0 \qquad \Rightarrow y = 0.$$

$$x = 0$$

Critical point $(0,0)$

$$f_{xx} = 2e^{-x^2-y^2}(2x^2-1), \quad f_{yy} = 2e^{-x^2-y^2}(2y^2-1)$$

$$f_{xy} = 4xy\, e^{-x^2-y^2} = f_{yx}.$$

$$D(x,y) = \left[2e^{-x^2-y^2}(2x^2-1)\right] \cdot \left[2e^{-x^2-y^2}(2y^2-1)\right] - \left[4xye^{-x^2-y^2}\right]$$

$$D(0,0) = [2(-1)] \cdot [2(-1)] - 0 = 4 > 0$$

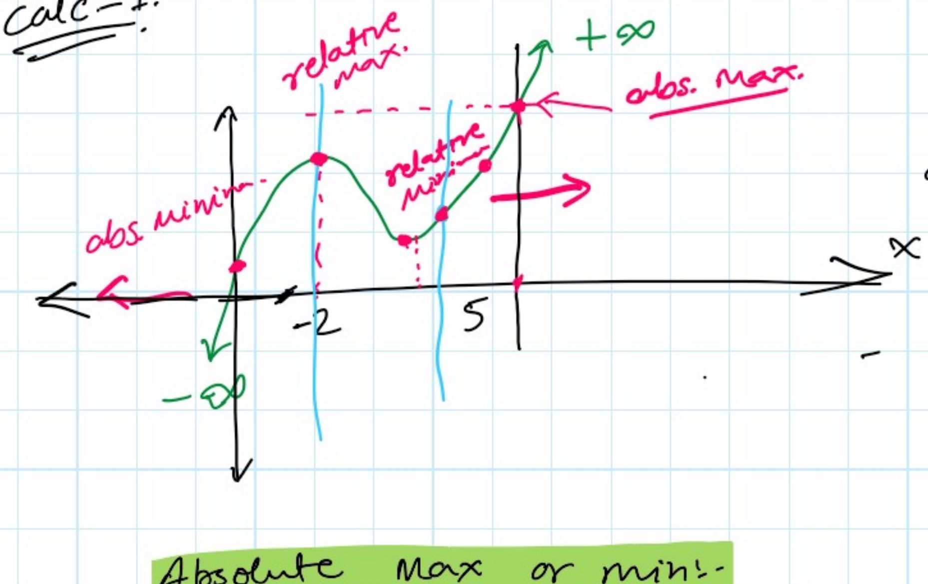
$$\& \ f_{xx}(0,0) = -2 < 0$$

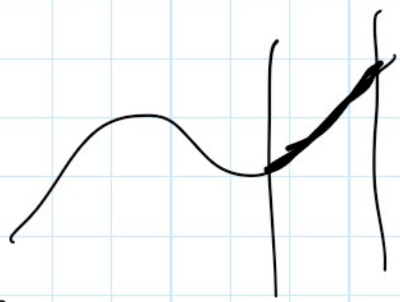
concave down

Relative Maxima at $(0,0)$ value of $f(0,0) = 1$.

Have we discussed abs. Minima & Maxima.

Calc-1.





Absolute Max or min:-

Calc-1:
$y=f(x)$

It will have relative max or min in open domain (that we discuss), but if we provide the fixed domain $[a,b]$ (may be $[-4,7]$)

→ This domain will give absolute max or min that occur at boundary or at critical points.

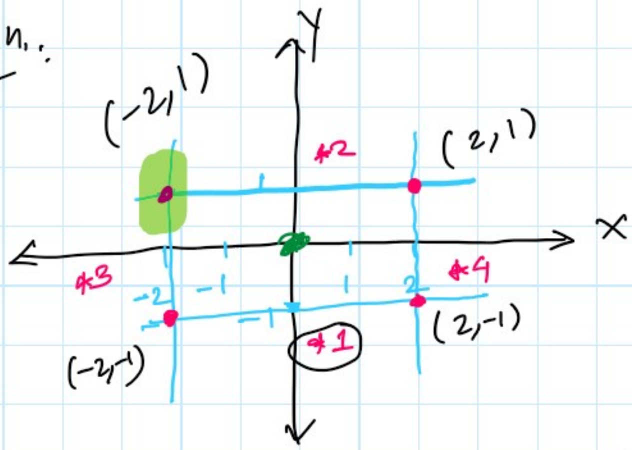
Calc-3:

For $f(x,y)$, A continuous function on a closed region. There must exist an absolute max or min.

and they must occur at either a critical points or on a boundary of the region.

Ex: $f(x,y) = x^2+xy+y^2$ on $R: \{(x,y) \mid -2 \le x \le 2, -1 \le y \le 1\}$ ✓

soln.:



$f_x = 2x + y = 0$

$f_y = x + 2y = 0$

$\boxed{y=1, x=-2}$

C.P. $(-2, 1)$

*1

$y=-1, x=x$

$f(x,-1) = x^2 - x + 1$

$f'(x,-1) = 2x - 1$

$= 0$ (for critical points)

$x = 1/2, y = -1$

C.P. $(\frac{1}{2}, -1) \to 3/4$

E.P. $(-2,-1) \to 7$

E.P. $(2,-1) \to 3$

*2 $y=1, x=x$

$f(x,1) = x^2+x+1$

$f'(x,1) = 2x+1$

$= 0$

$x = -1/2, y=1$

C.P. $(-\frac{1}{2}, 1) \to 3/4$

E.P. $(2,-1) \to 3$

E.P. $(2, 1) \to 7$

*4 $x=2, y=y$

$z = f(x,y)$

E.P. (2, ...

$z = f(x,y)$

*3) $x=-2,\ y=y$

$f(-2,y) = 4-2y+y^2$

$f'(-2,y) = -2+2y$
$= 0$

$y = 1,\ x=-2$

C.P. $(-2,1) \rightarrow 3$

E.P. $(-2,-1) \rightarrow 7$

EP $(-2,1) \rightarrow 3$

*4 $x=2,\ y=y$

$f(2,y) = 4+2y+y^2$

$f'(2,y) = 2+2y = 0$

$y=-1,\ x=2$

C.P. $(2,-1) \rightarrow 3$

E.P. $(2,-1) \rightarrow 3$

E.P. $(2,1) \rightarrow 7$

Absolute max $= 7$

Absolute min $= 0$

$(0,0) \Rightarrow z=0$