Directional derivative

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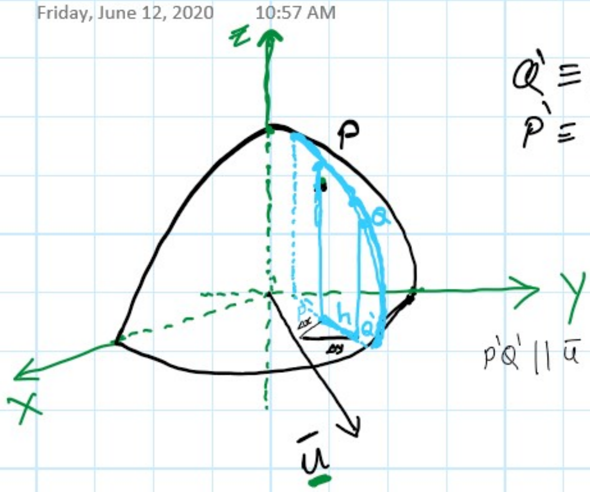
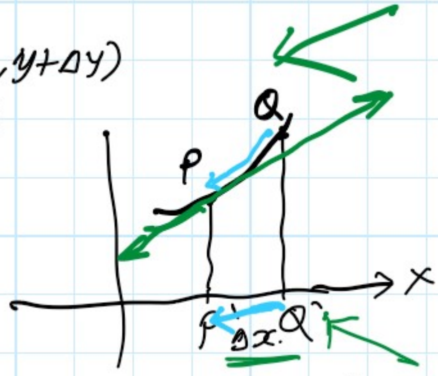
Directional derivative



$Q' \equiv (x+\Delta x, y+\Delta y)$
$P' \equiv (x, y)$



$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

To give a direction we must have a vector (line vector) on the plane of the independent variable.

$\Rightarrow$ the things we need: ① direction $\hat{u}$
② Need a point

$\to$ make a line through $P' \parallel \hat{u}$ to create a plane $\parallel$ to $\hat{u}z$ plane.

$\Rightarrow$ PQ is a secant line :-
As $Q' \to P'$, the secant approaches to the tangent at P.

So Δx & Δy must $\to 0$.

Now, $P'Q' \parallel \hat{u} \implies \overline{P'Q'} = h \cdot \hat{u}$

$$= h \cdot (u_1 \bar{i} + u_2 \bar{j})$$
$$= h u_1 \bar{i} + h u_2 \bar{j}$$

$\because$ $P'Q' = \Delta x\, \bar{i} + \Delta y\, \bar{j}$

$\Rightarrow$ $\Delta x = h u_1$ & $\Delta y = h u_2$

$$\text{Derivative} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$(\,)^2$

$\to$ $\Delta x^2 = h^2 u_1^2$ & $\Delta y^2 = h^2 u_2^2$

$\hookrightarrow \quad \Delta x^2 = h^2 u_1^2 \quad \& \quad \Delta y^2 = h^2 u_2^2$

Add $\Rightarrow \Delta x^2 + \Delta y^2 = h^2 u_1^2 + h^2 u_2^2$

$$= h^2 (u_1^2 + u_2^2)$$

$\sqrt{\ } \Rightarrow \sqrt{\Delta x^2 + \Delta y^2} = h \sqrt{u_1^2 + u_2^2} = 1.$

$$h = \sqrt{\Delta x^2 + \Delta y^2}$$

$$D_{\hat{u}}(x,y) = f_x \cdot u_1 + f_y \cdot u_2$$

For $\hat{u}$ is unit vector.

$$= f_x \cdot \underline{dx} + f_y \underline{dy}.$$

$$= f_x \underline{\Delta x} + f_y \underline{\Delta y}$$

$$= f_x \underline{\underline{u_1 + f_y u_2}}$$

Ex Using definition, find the derivative of $f(x,y) = x^2 + xy$ at $P_0(1,2)$ in the direction of the unit vector $u = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$

Sol^n $\quad \|\hat{u}\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

$$Df_{\hat{u}}(x,y) = \lim_{h \to 0} \frac{f(x_0 + \overbrace{h u_1}^{\Delta x}, y_0 + \overbrace{h u_2}^{\Delta y}) - f(x_0, y_0)}{h}$$

$$= \lim_{h \to 0} \frac{f\left(1 + h \times \frac{1}{\sqrt{2}}, 2 + h \times \frac{1}{\sqrt{2}}\right) - f(1,2)}{h}$$

$$= \lim_{h \to 0} \frac{\left(1 + \frac{h}{\sqrt{2}}\right)^2 + \left(1 + \frac{h}{\sqrt{2}}\right)\left(2 + \frac{h}{\sqrt{2}}\right) - 3}{h}$$

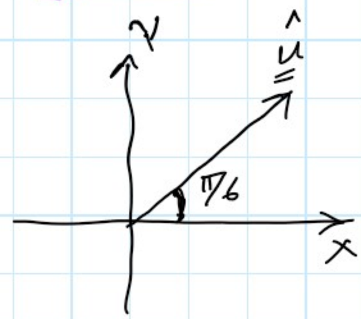
$$= \lim_{h \to 0} \frac{\cancel{1} + \frac{2h}{\sqrt{2}} + \frac{h^2}{2} + \cancel{2} + \frac{3h}{\sqrt{2}} + \frac{h^2}{2} \cancel{- 3}}{h}$$

$$\frac{5h}{\ } + h^2$$

$$= \lim_{h \to 0} \frac{\frac{5h}{\sqrt{2}} + h^2}{h} = \frac{5}{\sqrt{2}}$$

Ex! Find the derivative of $f(x,y) = x^3 - 2x^2 + y^3$ at $P(1,2)$ in the direction of the vector that makes an angle of $\theta = \pi/6$ with x-axis.

Sol^n: → what is directional vector (unit vector)
→ point $(1,2)$



$$\hat{u} = \cos\frac{\pi}{6}\, i + \sin\frac{\pi}{6}\, j$$
$$\hat{u} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

$\hat{u} = \cos\frac{\pi}{6}\hat{i} + \sin\frac{\pi}{6}\hat{j}$

→ $\hat{u} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$

$\because \|\hat{u}\| = 1.$

$f_x = 3x^2 - 4x.$

$f_y = 3y^2$

$$D_{\hat{u}}(x,y) = (3x^2 - 4x)\cdot\frac{\sqrt{3}}{2} + (3y^2)\frac{1}{2} \leftarrow$$

Equation of slope of tangent line to $f(x,y)$, a surface at any point but only in direction of $\hat{u}$.

$$D\hat{u}(1,2) = (3-4)\cdot\frac{\sqrt{3}}{2} + 12 \times \frac{1}{2}$$
$$= -\frac{\sqrt{3}}{2} + 6$$

$$D_{\hat{u}} f(x,y) = f_x \cdot u_1 + f_y \cdot u_2 \qquad \hat{u} = u_1\hat{i} + u_2\hat{j}$$

Scalar

vector.

$$= \begin{pmatrix} f_x \\ f_y \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

vector. unit vector.

$$= \nabla f(x,y) \cdot \hat{u} \qquad \text{(Dot)}$$

~~vector~~ $= \nabla f(x,y) \cdot \hat{u}$ ((Dot))

vector   vector. || Scalar.

Gradient vector. $(\nabla f_{(x,y)})$

→ $D_{\hat{u}}$ is a slope of surface Not a vector.

→ ∇f is a vector (part of $D_{\hat{u}}$)

$$D_{\hat{u}} f(x,y) = \nabla f(x,y) \cdot \hat{u}$$

→ $\nabla f(x,y)$ Relates to Grade (climb) of the surface (for a specific grade, must have a point & $\hat{u}$)

Property of ∇f:

1) If $\nabla f = \bar{0}$, then $D_{\hat{u}} f = 0$ for any $\hat{u}$

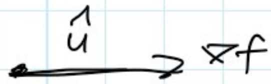
$$\hat{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

2) $D_{\hat{u}} f(x,y)$ has its maximum value of $\|\nabla f(x,y)\|$

$$D_{\hat{u}} f(x,y) = \nabla f(x,y) \cdot \hat{u} \quad (1)$$

$=$

3) $D_{\hat{u}} f(x,y) = \|\nabla f\| \cdot \|\hat{u}\| \cos\theta$



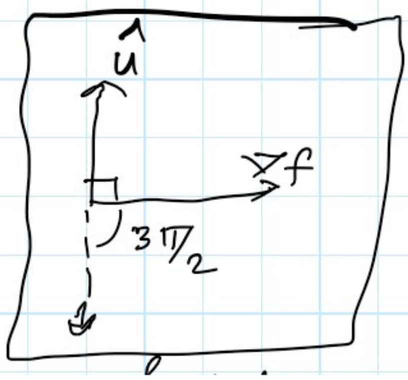
$$D_{\hat{u}} f(x,y) = \|\nabla f\| \cos\theta$$

$= \|\nabla f\|$ ← Max.

$= -\|\nabla f\|$ ← Min

$= 0 \Rightarrow \cos\theta = 0$

$\theta = \pi/2,\ 3\pi/2$

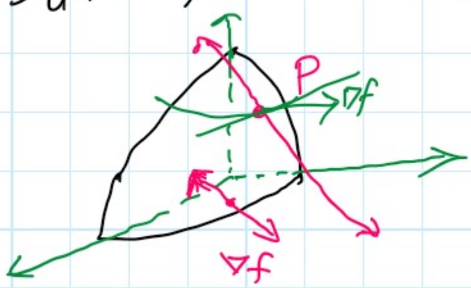


Any other θ gives a value less than 1 & takes a fraction of $\nabla f \Rightarrow D_{\hat{u}}$ become less steeper.

$D_{\hat{u}} f(x,y)$ has its minimum value of $-\|\nabla f(x,y)\|$ at $\theta = \pi$

So ∇f gives the vector for the steepest grade of a surface at a point.

If $\hat{u}$ is not $\parallel$ to ∇f, means $\hat{u}$ as turning $D_{\hat{u}} f$ from the direction of steepest climb. ∇f



$\bar{a} = \langle 5, 6 \rangle$ $|\bar{a}|$

$\bar{b} = \langle -6, 5 \rangle$ or $\bar{b} = \langle 6, -5 \rangle$ $|\bar{b}|$

$\bar{a} \cdot \bar{b} = 0$

0

③ $D_{\hat{u}} f = \langle 60, -60 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle$

Now $D_{\hat{u}} f = -60$