Chain Rule

Monday, June 8, 2020 11:10 AM

$$\Delta z \approx dz = f_x\, dx + f_y\, dy.$$

partial derivative.

Q) Pressure of a certain gas can be described by $P = \dfrac{8.314\,T}{V}$, find dP,

If volume goes from $20L \to 20.2L$ & temperature from $300K \to 295K$. find approximate change in pressure.

Soln:- $\boxed{dP = P_T\, dT + P_V\, dV} \Leftarrow$ $\boxed{\dfrac{dP}{P} \times 100}$

$$\begin{cases} dT = \Delta T = -5 \\ dV = \Delta V = 0.2 \end{cases} \qquad \begin{cases} T = 300K \\ V = 20L \end{cases}$$

$$P_V = \frac{-8.314}{V^2} T, \quad P_T = \frac{8.314}{V}$$

$$\Delta P \approx dP = -3.3256 \; P_A$$

$1L = 1dm^3$

$1m^3$

$$dP = \frac{8.314}{20} \times (-5) + \left(\frac{-8.314}{400}\right) \times (0.2) \times 300$$

$=$

Actual use of Differential:-

⇒ To find the max. error in the function while measuring the independent variable.

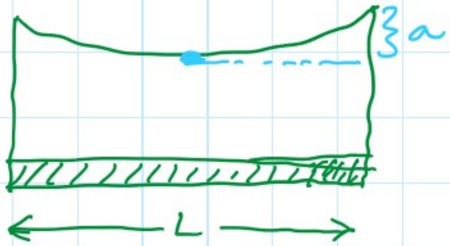
→ Error may be given in terms of %.

⇒ Error may be given in terms of %.

$$\text{max \% error} = \left|\frac{\Delta f}{f}\right| \approx \left|\frac{df}{f}\right|$$

Ex Tension T of a cable at its lowest point is given by

$$T = \frac{wL^2}{8a}$$



w = weight of road in lb/ft

% error is

$$\left.\begin{array}{l} w \to \pm 1\% \\ L \to \pm 2\% \\ a \to \pm 2\% \end{array}\right\} \to \left|\frac{dw}{w}\right| = 1\% = 0.01, \quad \left|\frac{dL}{L}\right| = 0.02 \ \& \ \left|\frac{da}{a}\right| = 0.02$$

Find % error in T.

Soln:

$$\left|\frac{dT}{T}\right| = 0.07 \quad \text{or } 7\%$$

$$dT = T_l\, dl + T_a\, da + T_w\, dw$$

$$\Delta T \approx dT = \frac{L^2}{8a} dw + \frac{2w}{8a} L\, dL - \frac{wL^2}{8a^2} da$$

Divide by T

$$\left|\frac{dT}{T}\right| = \left| \frac{\frac{L^2}{8a} dw}{\frac{wL^2}{8a}} + \frac{\frac{2w}{8a} L}{\frac{wL^2}{8a}} dL - \frac{\frac{wL^2}{8a^2}}{\frac{wL^2}{8a}} da \right|$$

$$= \left|\frac{dw}{w}\right| + \left|2 \cdot \frac{dL}{L}\right| + \left|-\frac{da}{a}\right|$$

$$= 0.01 + 2 \times (0.02) + 0.02$$

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$= 0.07$ or 7%

CHAIN RULE:-

$$y = [\sin(x^5)]^5 \Longleftarrow$$

$y = \sin(x^5)$

$u = x^5$

$y = \sin u$

u - intermediate variable.

$\frac{dy}{dx} = ?$

3 variable.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dt} \cdot \frac{dt}{dx}.$$

Chain Rule in multivariable f^n:-

$$w = x^2 - y^2 \quad , \quad x = t^2 + 1 \quad , \quad y = t^3 + t$$

$\frac{dw}{dt} = ?$ x & y are intermediate variable & t is the independent variable.

Proof: As t changes (Δt), x & y respond by changing also (Δx & Δy) & both Δx & Δy cause a change in w (Δw)

$$\Delta w = \left(\begin{matrix}\text{The Amount } x \\ \text{changes } w\end{matrix}\right) + \left(\begin{matrix}\text{The amount } y \\ \text{changes } w\end{matrix}\right)$$

$$\boxed{m = \frac{\text{Rise}}{\text{Run}}}$$

$$\Delta w = \frac{\partial w}{\partial x} \cdot \Delta x + \frac{\partial w}{\partial y} \cdot \Delta y$$

Divide by Δt

$$\frac{\Delta w}{\Delta t} = \frac{\partial w}{\partial x} \cdot \frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y} \cdot \frac{\Delta y}{\Delta t}$$

$$\lim \frac{\Delta w}{} = \lim \left(\frac{\partial w}{}\right) \frac{\Delta x}{} + \lim_{\Delta t \to 0} \frac{\partial w}{\partial y} \cdot \frac{\Delta y}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\partial w}{\partial x}\right) \frac{\Delta x}{\Delta t} + \lim_{\Delta t \to 0} \frac{\partial w}{\partial y} \cdot \frac{\Delta y}{\Delta t}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y} \cdot \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}} \Leftarrow$$

Ex: $w = x^2 - y^2, \quad x = t^2 + 1, \quad y = t^2 + t$

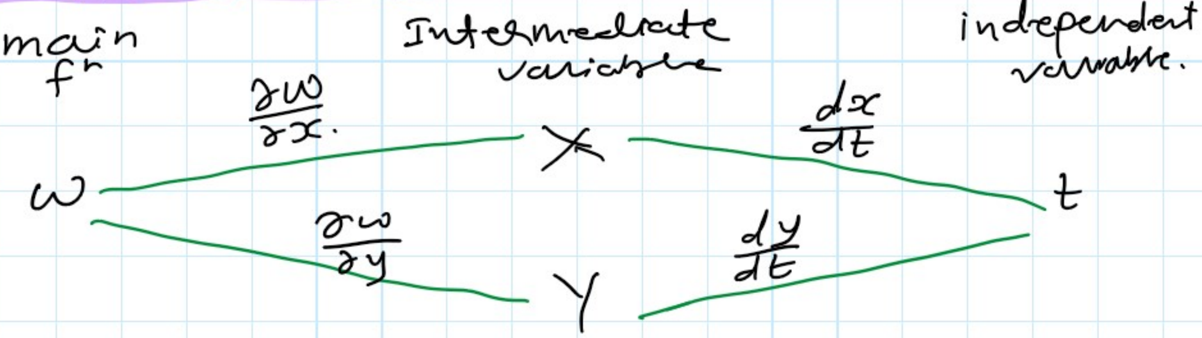
$$\frac{\partial w}{\partial x} = 2x \quad ; \quad \frac{dx}{dt} = 2t \quad , \quad \frac{dy}{dt} = 2t + 1$$

$$\frac{\partial w}{\partial y} = -2y \quad ;$$

$$\frac{dw}{dt} = 2x \cdot 2t + (-2y)(2t + 1)$$

$$= 4xt - 2y(2t + 1)$$

Summarize the Idea:-



$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

Q. $w = x^2 y - x y^3 \quad ; \quad x = \cos t \quad , y = e^t$

Find $\left.\frac{dw}{dt}\right|_{t=0}$

Soln.

for $t = 0$, $x = 1$
$y = 1$.

$$\frac{dw}{dt} = -2$$

$y=1.$

$$\boxed{\frac{dw}{dt} = -2}$$

$$\frac{dw}{dt} = (2xy - y^3)(-\sin t) + (x^2 - 3xy^2)e^t$$

$$\left.\frac{dw}{dt}\right|_{\substack{t=0\\x=1\\y=1}} = -2$$

Extending the concept:-

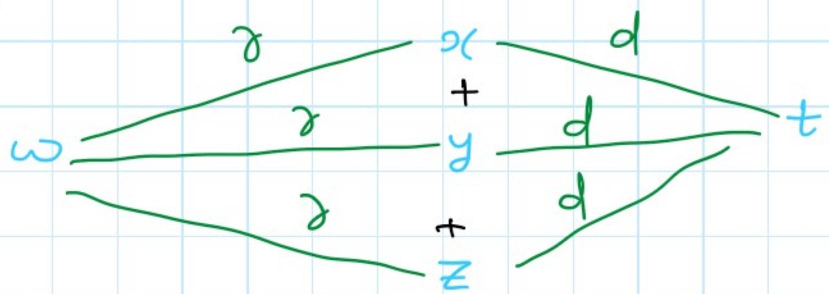
A) more intermediate variable.

B) More independent variable.

(A) More intermediate variable:-

$$w = \tan^{-1}(xz) + \frac{z}{y} \quad , x = t, \; y = t^2, \; z = \sin t$$

$$\frac{dw}{dt} = ?$$



$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

$$\frac{dw}{dt} = \left(\frac{z}{1 + x^2 z^2}\right) - \frac{2zt}{y^2} + \left(\frac{x}{1+x^2z^2} + \frac{1}{y}\right)\cos t$$