Linear combination of two independent variable

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1) $E(ax+b) = a\,E(x) + b.$

2) $var(ax+b) = a^2\,var(x)$

$s = ax \pm by$

$E(s) = E(ax \pm by) = a\,E(x) \pm b\,E(y)$ — ①

$var(ax \pm by) = a^2\,var(x) + b^2\,var(y)$ — ②

$(-b)^2 = b^2$

Roll a die:-

x_i	1	2	3	4	5	6
$p(x)$	1/6	1/6	1/6	1/6	1/6	1/6

$E(x) = \Sigma x_i \cdot p = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$

$= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{21}{6}$

$E(x) = \mu = 3.5$ (Hypothetical value).

$var(x) = E(x^2) - (E(x))^2$

$= \Sigma x^2 \cdot p - [E(x)]^2 = \frac{35}{12}$

Rolling 2 dice: $n(s) = 36$

S	2	3	4	5	6	7	8	9	10	11	12
$p(s)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36.

S	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8

$E(S) = \frac{2}{36} + 3\times\frac{2}{36} + 4\times\frac{3}{36} + \text{----}$

1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$$E(S)=\frac{\ldots}{36}+\frac{\ldots}{36}+\frac{\ldots}{36}+\ldots\ldots + 12\times\frac{1}{36}$$

$$=\frac{2+6+12+20+30+42+40+36+30+22+12}{36}$$

$$=252/36=7$$

verify.

$$E(S)=E(X+X)=E(X)+E(X)$$

$$=3.5+3.5=7$$

Ex

If X is a random variable with $E(X)=25$, $Var(X)=8$, and Y is a random variable with $E(Y)=15$, $Var(Y)=6$, and X and Y are independent, find:

i) $E(X+Y)$ and $Var(X+Y)$

ii) $E(X-Y)$ and $Var(X-Y)$.

(i) $E(X+Y)=E(X)+E(Y)=25+15=40$

$Var(X+Y)=Var(X)+Var(Y)=8+6=14$

(ii) $E(X-Y)=25-15=10$

$Var(X-Y)=8+6=14$

Ex

A fair coin is tossed and a fair die is thrown. Y is the score showing on the die, and $X=0$ if the coin lands heads and $X=1$ if the coin lands tails.

i) Find the expectation and variance of each of X and Y.

ii) Find the probability distribution of $X+Y$.

iii) Use the probability distribution in ii) to find the expectation and variance of $X+Y$.

iv) Show that $E(X+Y)=E(X)+E(Y)$ and $Var(X+Y)=Var(X)+Var(Y)$.

①

X	0	1
p	1/2	1/2

②

Y	1	2	3	4	5	6
p	1/6	1/6	1/6	1/6	1/6	1/6

$E(X)=\Sigma xp=0\times\frac{1}{2}+1\times\frac{1}{2}$ $\quad E(Y)=\frac{7}{\ldots}$

$E(X) = \sum x \cdot p = 0 \times \frac{1}{2} + 1 \times \frac{1}{2}$
$= 1/2$

$E(Y) = \frac{7}{2}$

$E(X^2) = \sum x^2 \cdot p = 0^2 \cdot \left(\frac{1}{2}\right) + 1^2 \cdot \left(\frac{1}{2}\right)$
$= \frac{1}{2}$

$\operatorname{var}(X) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$\operatorname{var}(Y) = \frac{35}{12}$

(ii) $Z = X + Y$

Z	1	2	3	4	5	6	7
p.	1/12	2/12	2/12	2/12	2/12	2/12	1/12

(*)

X\Y	1	2	3	4	5	6
0	1	2	3	4	5	6
1	2	3	4	5	6	7

$6 \times 2 = 12$
outcomes

$E(Z) = \sum z \cdot p$.
$\operatorname{Var}(Z) = E(Z^2) - [E(Z)]^2 =$

$E(Z) = E(X+Y) = E(X) + E(Y) = \frac{1}{2} + \frac{7}{2} = 4$

$\operatorname{Var}(Z) = \operatorname{Var}(X+Y) = \operatorname{Var}(X) + \operatorname{Var}(Y) = \frac{1}{4} + \frac{35}{12}$
$= \frac{38}{12} = \frac{19}{6}$

Expectation & variance of mean :- $(\bar{x})$

$$\bar{x} = \frac{\sum x_i}{n}$$
(Mean)

If $x_1, x_2, x_3, ---x_n$ are independent observation of a random variable X.

$E(X) = \mu$, $\operatorname{var}(X) = \sigma^2$ (σ : standard deviation)

$$E(\bar{x}) = E\left(\frac{1}{n}\sum x_i\right) = \frac{1}{n} E\left(\sum x_i\right)$$

$$= \frac{1}{n}\left\{E(x_1) + E(x_2) + E(x_3) + ---- + E(x_n)\right\}$$

$$= \frac{1}{n}\left\{\mu + \mu + \mu + ----- + \mu\right\}$$

$$= \frac{1}{n}\, n \cdot \mu = \mu$$

$$\boxed{E(\bar{x}) = \mu}$$ ①

$$\operatorname{var}(\bar{x}) = \operatorname{var}\left(\frac{1}{n}\sum x_i\right) = \frac{1}{n^2}\operatorname{var}\left(\sum x_i\right)$$

$$= \frac{1}{n^2}\left[\operatorname{var}(x_1) + \operatorname{var}(x_2) + --- + \operatorname{var}(x_n)\right]$$

$$= \frac{1}{n^2}\left[\sigma^2 + \sigma^2 + --- \text{ n times}\right]$$

$$= \frac{1}{n^2}\left[n\sigma^2\right] = \frac{\sigma^2}{n}$$

$$= \frac{1}{n^2}\left[n\sigma^2 \right] = \frac{\sigma^2}{n}$$

$$\boxed{Var(\bar{x}) = \frac{\sigma^2}{n}}$$

$Y = ax.$

$var(Y) = var(ax)$
$= a^2 var(x)$

Ex

X is a random variable with $E(X) = 24$, $Var(X) = 8$. Find the expectation and variance of

i) $\sum_{i=1}^{5} X_i$ ii) $\bar{X}_{12} = \frac{1}{12}\sum_{i=1}^{12} X_i$

(i) $E\left(\sum_{i=1}^{5} x_i\right) = E(x_1 + x_2 + x_3 + x_4 + x_5)$

$= E(x_1) + E(x_2) + E(x_3) + E(x_4) + E(x_5)$

$= 24 + 24 + 24 + 24 + 24 = 120$

$\sum_{i=1}^{5} x_i = (x_1 + x_2 + x_3 + x_4 + x_5)$

$i = 1, 2, 3, 4, 5$

$Var\left(\sum_{i=1}^{5} x_i\right) = Var(x_1 + x_2 + x_3 + x_4 + x_5)$

$= var(x_1) + var(x_2) + var(x_3) + \dots + var(x_5)$

$= 5 \times 8 = 40$