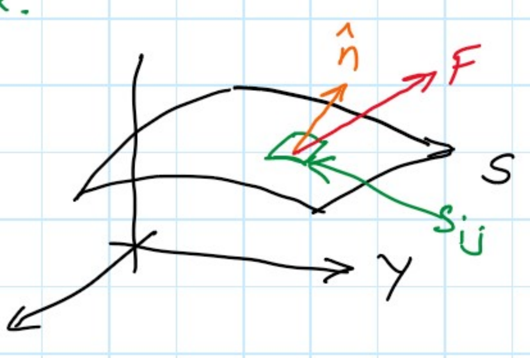
Divergence and Stokes' Theorem

Saturday, September 5, 2020 9:14 AM

If 'F' A vector field, contains a surface 'S' then 'F' describes the velocity of flow/field at any point across the surface.

- The rate of flow (amount/volume of flow across normal to the surface) is called the flux.



vector field · Normal at any point on surface × Really small piece of surface (area)

The surface integral of F over S is

$$\iint_S F \cdot dS = \iint_S F \cdot n \, dS$$ called a flux integral.

If fluid has a mass density at a point $\rho(x,y,z)$

$$\iint_S \rho \cdot F \cdot \hat{n} \, dS$$ mass of fluid flowing across S

For $F = P\hat{i} + Q\hat{j} + R\hat{k}$

$$\boxed{\iint F \cdot dS = \iint F \cdot \hat{n} \, dS = \iint (-Pg_x - Qg_y + R) \, dA}$$

$$\boxed{\iint_S F\cdot dS = \iint_S F\cdot \hat{n}\, dS = \iint_D \left(-Pg_x - Qg_y + R\right) dA}$$

for surface $Z = g(x,y)$ — projection of $z = g(x,y)$ onto xy-plane

$x = x,\quad y = y,\quad z = g(x,y)$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial g}{\partial x} \\ 0 & 1 & \frac{\partial g}{\partial y} \end{vmatrix} = -\frac{\partial g}{\partial u}\,\hat{i} - \frac{\partial g}{\partial v}\,\hat{j} + \hat{k}$$

$$\iint_S dS = \iint \|\vec{r_u} \times \vec{r_v}\|\, dA$$

$S \longrightarrow D$

Ex Find Flux of 'F' across surface bound by $Z = 1 - x^2 - y^2$ & $x^2 + y^2 = 1$ where Flow is described by $F = \overset{P}{x}\,\hat{i} + \overset{Q}{y}\,\hat{j} + \overset{R}{Z}\,\hat{k}$

Solⁿ.

$$\text{Flux} = \iint_S F\cdot\hat{n}\, dS = \iint_D \left(-Pg_x - Qg_y + R\right) dA$$

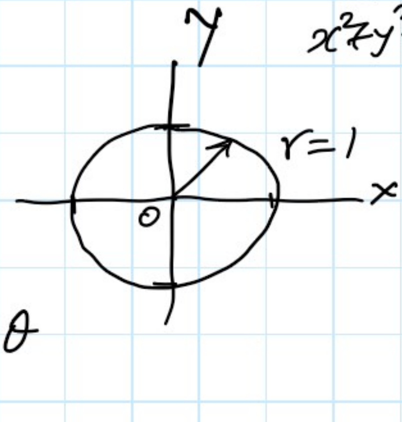
Let $Z = 1 - x^2 - y^2$

$g(x,y) = 1 - x^2 - y^2, \quad g_x = -2x, \quad g_y = -2y$

$$\text{Flux} = \iint_D \left[(-x)(-2x) - y(-2y) + Z\right] dA$$

$Z = 0$

$x^2 + y^2 = 1$



$$= \iint_D 2x^2 + 2y^2 + Z\, dA$$

$$= \iint_D \left[2(x^2 + y^2) + Z\right] r\, dr\, d\theta$$

$$= \iint_D \left[2(x^2+y^2)+z\right]$$

$$= \iint_D \left[2(x^2+y^2) + 1-(x^2+y^2)\right] r\,dr\,d\theta$$

$$= \iint_D \left[x^2+y^2+1\right] r\,dr\,d\theta.$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^3 + r \quad dr\,d\theta$$

$$= \boxed{\frac{3\pi}{2}}$$

If 'S' is simple closed surface:-

$$\text{Flux} = \iint_S F\cdot\hat{n}\, ds = \iiint_T \text{Div}\,F \; dV$$

$$\text{Div}\,F = \nabla\cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Divergence theorem:-

Ex Find the flux through vector field

$$F = (x+\sin z)\,\hat{i} + (2y+\cos x)\,\hat{j} + (3z+\tan y)\,\hat{k}$$

over the surface $x^2+y^2+z^2=4$ Sphere.

$\text{Div}F = 1+2+3 = 6.$

$$\text{Flux} = \iiint \text{Div}\,F\, dV.$$

$$\text{Div} F = 1+2+3=6.$$

$$\text{Flux} = \iiint_T \text{Div } F \, dV.$$

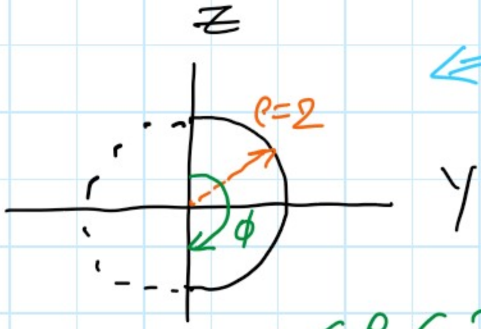
$$= \iiint_T 6 \, dV.$$

$$\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

$$x^2+y^2+z^2=\rho^2$$

$$\rho^2 = 4$$

$$\rho = 2$$



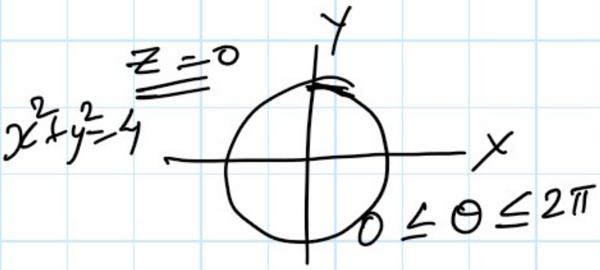
$$\Leftarrow x^2+y^2+z^2=4$$

$$x=0$$

$$y^2+z^2=4$$

$$0 \le \rho \le 2$$

$$0 \le \phi \le \pi$$



$$\text{Flux} = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{2} 6 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

$$= \int_0^{2\pi}\int_0^{\pi} 2 \, [\rho^3]_{\rho=0}^{\rho=2} \sin\phi \, d\phi \, d\theta.$$

$$= \int_0^{2\pi}\int_0^{\pi} 16 \sin\phi \, d\phi \, d\theta.$$

$$= \int_0^{2\pi} -16 \, [\cos\phi]_0^{\pi} \, d\theta = \int_0^{2\pi} -16(-1-1) \, d\theta$$

$$= 32\int_0^{2\pi} d\theta = 32\,[\theta]_0^{2\pi} = 64\pi$$

Ex Flux of $F = xy^2\,\hat{i} + yz^2\,\hat{j} + zx^2\,\hat{k}$

across surface $x^2+y^2=4$, $z=0$, $z=5$ (cylinder)

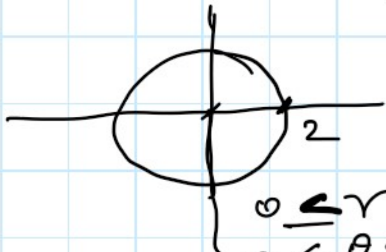
$$\text{Flux} = \iint_S F\cdot\hat{n}\,ds = \iiint_T \operatorname{Div} F\, dv.$$

$$\operatorname{Div} F = y^2+z^2+x^2$$

$$\text{Flux} = \iiint_T x^2+y^2+z^2\, dV.$$

z-simple. $0 \le z \le 5$

R on xyplane



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{5} (x^2+y^2+z^2)\, r\, dz\, dr\, d\theta$$

$$= \int_{0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{5} (r^2+z^2)\, r\, dz\, dr\, d\theta.$$

$$= \int_{0}^{2\pi}\int_{0}^{2}\int_{z=0}^{5} r^3 + r z^2\, dz\, dr\, d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{2} \left[r^3 z + \frac{1}{3} r z^3 \right]_0^5 dr\, d\theta.$$

$$= \int_{0}^{2\pi}\int_{0}^{2} \left[5r^3 + \frac{125}{3} r \right] dr\, d\theta.$$

$$= \int_0 \int_0 [\quad]$$

$$= \int_0^{2\pi} \left[\frac{5}{4} r^4 + \frac{125}{6} r^2 \right]_{r=0}^{r=2} d\theta .$$

$$= \int_0^{2\pi} 20 + \frac{250}{3} \, d\theta = \int_0^{2\pi} \frac{310}{3} \, d\theta$$

$$= \frac{310}{3} [\theta]_0^{2\pi} = \frac{620}{3} \pi$$

Stokes' Thm:-

$$W = \oint_C F \cdot dr = \iint_S \overbrace{\text{curl} F}^{\text{"F"}} \cdot ds .$$

'C' Not on a plane.

→ work done along 'c' through 'F' equals the Flux of the curl of "F" across any surface where 'c' is the boundary.

→ consider curl F as "F" a new vector field.

$$W = \oint_C F \cdot dr = \iint_S \text{curl} F \cdot ds = \iint_S \text{"F"} \, ds$$

$$= \iiint_T \text{Div "F"} \, dV .$$

for 'S' a simple closed surface

or $\iint_D (-Pg_x - Qg_y + R)\, dA$.

'S' is a surface.
$z = g(x,y)$

Ex $\oint_C F \cdot dr$, $F(x,y,z) = \underset{P}{\cos z}\, \hat{i} + \underset{Q}{x^2}\, \hat{j} + \underset{R}{2y}\, \hat{k}$

'C' the intersection of plane.
$z = 2 - x$ & $x^2 + y^2 = 4$.

$$\oint_C F \cdot dr = \iint_S \text{curl} F \cdot dS$$

$$\underline{\text{curl} F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos z & x^2 & 2y \end{vmatrix} = (2-0)\hat{i} - (0 + \sin z)\hat{j} + 2x\hat{k}$$

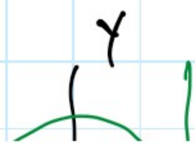
$$= 2\hat{i} - \sin z\, \hat{j} + 2x \hat{k}$$

$$"F" = 2\hat{i} - \sin z\, \hat{j} + 2x\hat{k}$$

$$\iint_S "F" \cdot dS = \iint_D -"P" g_x - "Q" g_y + "R"\, dA.$$

$z = g(x,y) \Rightarrow z = 2 - x$
$g_x = -1$, $g_y = 0$

$$= \iint_D (-2)(-1) - (-\sin z)(0) + 2x\, dA$$



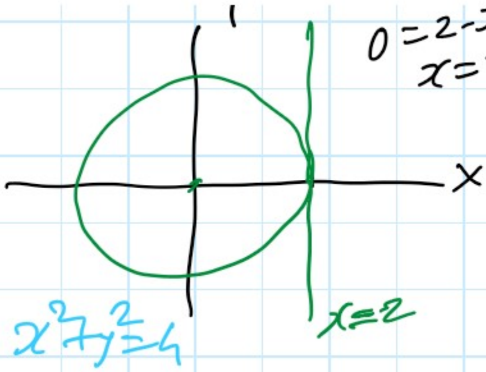
$z = 2 - x$.
$0 = 2 - x$
$x = 2$

D

$$= \iint_D 2 + 2x \; dA.$$

$0 = 2 - x$

$x = 2$



$x = r\cos\theta$

$y = r\sin\theta.$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} \left(2 + 2r\cos\theta\right) r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} 2r + 2r^2\cos\theta \; dr \, d\theta$$

$$= \int_0^{2\pi} \left[r^2 + \frac{2}{3} r^3 \cos\theta \right]_{r=0}^{r=2} d\theta.$$

$$= \int_0^{2} \left[4 + \frac{16}{3}\cos\theta\right] d\theta = \left[4\theta + \frac{16}{3}\sin\theta\right]_0^{2\pi}$$

$$= 8\pi$$

work done.

THE END.