**Q1. [2013/Prelim/MI/II/3(iii)(modified)]**

The function f is defined by

$$f : x \to \frac{x-2}{1+2x}, \quad x \in \mathbb{R},\ a < x < b.$$

Expand $f(x)$ in ascending powers of $x$ up to and including the term in $x^3$. State the minimum value of $a$ and the maximum value of $b$ for the expansion to be valid. **[3]**

**Solution**

$$\begin{aligned} f(x) &= (x-2)(1+2x)^{-1} \\ &= (x-2)(1-2x+4x^2-8x^3+\ldots) \\ &= -2+x+4x-2x^2-8x^2+4x^3+16x^3+\ldots \\ &= -2+5x-10x^2+20x^3\ldots \end{aligned}$$

For expansion to be valid,

$$|2x| < 1$$

$$|x| < \frac{1}{2}$$

$$-\frac{1}{2} < x < \frac{1}{2}$$

$\therefore$ minimum $a = -\frac{1}{2}$ and maximum $b = \frac{1}{2}$

**Q2. [2013/Prelim/RVHS/I/2]**

The series expansion of $(1+ax)^b$ up to and including the term in $x^2$ is given by $1-\frac{3}{2}x-\frac{3}{8}x^2$. Find the values of $a$ and $b$. Explain why the substitution $x=-2$ may not be suitable in estimating the value of $5^{\frac{7}{4}}$ using the above series. **[5]**

**Solution**

$$(1+ax)^b = 1+b(ax)+\frac{b(b-1)}{2!}(ax)^2+\ldots$$

$$=1+abx+\frac{a^2b(b-1)}{2}x^2+\ldots$$

Comparing coefficients,

$$ab=-\frac{3}{2} \Rightarrow a=-\frac{3}{2b} \quad \cdots (1)$$

$$\frac{a^2b(b-1)}{2}=-\frac{3}{8} \quad \cdots (2)$$

Substitute (1) into (2):

$$\left(-\frac{3}{2b}\right)^2\frac{b(b-1)}{2}=-\frac{3}{8}$$

$$\frac{3(b-1)}{b}=-1$$

$$3(b-1)=-b$$

$$b=\frac{3}{4}$$

$$\therefore a=-\frac{3}{2\left(\frac{3}{4}\right)}=-2$$

$$5^{\frac{7}{4}}=5\cdot 5^{\frac{3}{4}}=5[1+(-2)(-2)]^{\frac{3}{4}}$$

The series expansion is valid for

$$|ax|<1 \Rightarrow |-2x|<1 \Rightarrow -\frac{1}{2}<x<\frac{1}{2}$$

However, the substitution $x=-2$ falls outside the validity range. Thus, it is not suitable.

**Q1.** **[2013/Prelim/YJC/I/8]**

**(i)** Expand $\dfrac{\sqrt{4+x}}{1-x}$ in ascending powers of $x$, up to and including the term in $x^2$. State the set of values of $x$ for which the expansion is valid. **[5]**

**(ii)** By substituting $x=-\dfrac{1}{10}$, obtain an estimate for $\sqrt{390}$, leaving your answer as a fraction. **[2]**

**Solution**

**(i)**

$$\frac{\sqrt{4+x}}{1-x}=\sqrt{4}\left(1+\frac{x}{4}\right)^{\frac{1}{2}}(1-x)^{-1}$$

$$=2\left(1+\frac{1}{2}\left(\frac{x}{4}\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{x}{4}\right)^2+\cdots\right)\left(1+x+x^2+\cdots\right)$$

$$=2\left(1+\frac{x}{8}-\frac{x^2}{128}+\cdots\right)\left(1+x+x^2+\cdots\right)$$

$$=2\left(1+\frac{x}{8}-\frac{x^2}{128}+x+\frac{x^2}{8}+x^2\right)+\cdots$$

$$=2+\frac{9}{4}x+\frac{143}{64}x^2+\cdots$$

Valid for: $\left|\dfrac{x}{4}\right|<1\Rightarrow|x|<4$ and $|x|<1$.

Hence $\{x\in\mathbb{R}:-1<x<1\}$

**(ii)**

$$\frac{\sqrt{4-\frac{1}{10}}}{1+\frac{1}{10}}\approx 2+\frac{9}{4}\left(-\frac{1}{10}\right)+\frac{143}{64}\left(-\frac{1}{10}\right)^2$$

$$\sqrt{\frac{39}{10}}\approx\frac{11}{10}\times\frac{11503}{6400}$$

$$\frac{\sqrt{10}}{\sqrt{10}}\times\sqrt{\frac{39}{10}}\approx\frac{11}{10}\times\frac{11503}{6400}$$

$$\sqrt{390}\approx 10\times\frac{11}{10}\times\frac{11503}{6400}=\frac{126533}{6400}$$

**Q2. [2013/Prelim/IJC/I/8]**

Find the expansion of $\left(\frac{1+2x^2}{4-x}\right)^{\frac{1}{2}}$ in ascending powers of $x$, up to and including the term in $x^2$. **[4]**

**(i)** Find the set of values of $x$ for which the expansion is valid. **[2]**

**(ii)** By putting $x=\frac{1}{4}$, show that $\sqrt{30}\approx\frac{a}{b}$, where $a$ and $b$ are integers to be determined. **[2]**

**Solution**

$$\left(\frac{1+2x^2}{4-x}\right)^{\frac{1}{2}}=\left(1+2x^2\right)^{1/2}\left(4^{-1/2}\right)\left(1-\frac{x}{4}\right)^{-1/2}$$

$$=\frac{1}{2}\left(1+\frac{1}{2}\left(2x^2\right)+\ldots\right)\left[1-\frac{1}{2}\left(-\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-\frac{x}{4}\right)^2+\ldots\right]$$

$$=\frac{1}{2}\left(1+x^2+\ldots\right)\left(1+\frac{1}{8}x+\frac{3}{128}x^2+\ldots\right)$$

$$=\frac{1}{2}\left(1+\frac{1}{8}x+\frac{3}{128}x^2+x^2\right)+\ldots$$

$$=\frac{1}{2}\left(1+\frac{1}{8}x+\frac{131}{128}x^2\right)+\ldots$$

**(i)** $\left|2x^2\right|<1$ and $\left|\frac{x}{4}\right|<1$

$-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$ and $-4<x<4$

Taking intersection, the set of values is $\left\{x\in\mathbb{R}:-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}\right\}$.

**(ii)** $$\sqrt{\frac{1+2\left(\frac{1}{4}\right)^2}{4-\frac{1}{4}}}\approx\frac{1}{2}\left(1+\frac{1}{8}\left(\frac{1}{4}\right)+\frac{131}{128}\left(\frac{1}{4}\right)^2\right)$$

$\sqrt{\frac{3}{10}}\approx\frac{2243}{4096}$ $\left[\textbf{Note: }\sqrt{\frac{3}{10}}=\frac{\sqrt{3}\cdot\sqrt{10}}{\sqrt{10}\cdot\sqrt{10}}=\frac{\sqrt{30}}{10}\right]$

$\sqrt{30}\approx10\left(\frac{2243}{4096}\right)=\frac{11215}{2048}$

**Alternative**

$\sqrt{\frac{9}{30}}\approx\frac{2243}{4096}$

$\sqrt{30}\approx3\left(\frac{4096}{2243}\right)=\frac{12288}{2243}$

**Q3.** **[2013/Prelim/RI/I/1]** Integration

By considering the expansion of $\frac{1}{\sqrt{1-x^2}}$, or otherwise, show that

$$\sin^{-1} x = x + \frac{x^3}{6} + ax^5 + \dots$$

where $a$ is a constant to be determined. **[5]**

**Solution**

$$\frac{1}{\sqrt{1-x^2}} = \left(1-x^2\right)^{-\frac{1}{2}}$$

$$= 1 + \left(-\frac{1}{2}\right)\left(-x^2\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-x^2\right)^2 + \dots$$

$$= 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \dots$$

Since $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$, so

$$\sin^{-1} x = \int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$$

$$= \int \left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + \dots\right)\mathrm{d}x$$

$$= x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots + c, \text{ where } c \text{ is an arbitrary constant.}$$

When $x = 0$, $c = \sin^{-1} 0 = 0$.

Hence $\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots$, where $a = \frac{3}{40}$.