1

[Maximum mark: 6]

Solve the equation $z^3 = 1$, giving your answers in Cartesian form.

Solution

Using de Moivre's theorem, we have

$$\begin{aligned} z^3 &= 1\,\text{cis}(0) \\ &\Longrightarrow \\ z &= \sqrt[3]{1}\,\text{cis}\left(\frac{0+2n\pi}{3}\right) \quad \text{for} \quad n = 0, 1, 2. \end{aligned}$$

Hence we get

$$z_1 = \text{cis}(0) = 1,$$

$$z_2 = \text{cis}\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i},$$

$$z_3 = \text{cis}\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}.$$

2

[Maximum mark: 6]

Consider the complex number $z = \dfrac{w_1}{w_2}$ where $w_1 = \sqrt{2} + \sqrt{6}\text{i}$ and $w_2 = 3 + \sqrt{3}\text{i}$.

(a) Express w_1 and w_2 in modulus-argument form and write down

(i) the modulus of z;

(ii) the argument of z. [4]

(b) Find the smallest positive integer value of n such that z^n is a real number. [2]

Solution

(a) If we express w_1 and w_2 in modulus-argument form, then we get

$$w_1 = 2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{3}\right) \quad \text{and} \quad w_2 = 2\sqrt{3}\operatorname{cis}\left(\frac{\pi}{6}\right).$$

(i) The modulus of z is

$$\begin{aligned} |z| &= \frac{|w_1|}{|w_2|} \\ &= \frac{\cancel{2}\sqrt{2}}{\cancel{2}\sqrt{3}} \\ &= \frac{\sqrt{6}}{3}. \end{aligned}$$

(ii) The argument of z is

$$\begin{aligned} \arg z &= \arg w_1 - \arg w_2 \\ &= \frac{\pi}{3} - \frac{\pi}{6} \\ &= \frac{2\pi - \pi}{6} \\ &= \frac{\pi}{6}. \end{aligned}$$

(b) The smallest $n \in \mathbb{Z}^+$ such that z^n is a real number is given by

$$\begin{aligned} \sin\left(\arg\left(w^n\right)\right) &= 0 \\ \sin\left(n \arg w\right) &= 0 \\ \sin\left(\frac{n\pi}{6}\right) &= 0 \\ n &= 6. \end{aligned}$$

3

[Maximum mark: 7]

Consider the complex numbers $u = 1 + 2\mathrm{i}$ and $v = 2 + \mathrm{i}$.

(a) Given that $\frac{1}{u} + \frac{1}{v} = \frac{6\sqrt{2}}{w}$, express w in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{R}$. [4]

(b) Find w^* and express it in the form $re^{\mathrm{i}\theta}$. [3]

Solution

(a)

$$\begin{aligned}
\frac{6\sqrt{2}}{w} &= \frac{1}{u} + \frac{1}{v} \\
&= \frac{1}{1+2\mathrm{i}} + \frac{1}{2+\mathrm{i}} \\
&= \frac{1-2\mathrm{i}}{5} + \frac{2-\mathrm{i}}{5} \\
&= \frac{3-3\mathrm{i}}{5} \\
&\Longrightarrow \\
w &= \frac{30\sqrt{2}}{3-3\mathrm{i}} \\
&= \frac{30\sqrt{2}(3+3\mathrm{i})}{18} \\
&= 5\sqrt{2}(1+\mathrm{i}) \\
&= 5\sqrt{2} + 5\sqrt{2}\mathrm{i}.
\end{aligned}$$

(b)

$$\begin{aligned}
w^* &= 5\sqrt{2} - 5\sqrt{2}\mathrm{i} \\
&= 10e^{-\frac{\pi}{4}\mathrm{i}}
\end{aligned}$$

as required.

4

[Maximum mark: 9]

(a) Find three distinct roots of the equation $z^3 + 64 = 0$, $z \in \mathbb{C}$ giving your answers in modulus-argument form. [6]

The roots are represented by the vertices of a triangle in an Argand diagram.

(b) Show that the area of the triangle is $12\sqrt{3}$. [3]

Solution

(a) Using Euler's identity $-1 = \cos\pi + \mathrm{i}\sin\pi$, we get

$$\begin{aligned} z^3 &= -64 \\ &= 64(\cos\pi + \mathrm{i}\sin\pi). \end{aligned}$$

Hence, by de Moivre's theorem, we have

$$\begin{aligned} z &= \sqrt[3]{64}(\cos\pi + \mathrm{i}\sin\pi)^{\frac{1}{3}} \\ &= 4\left(\cos\left(\frac{\pi + 2n\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi + 2n\pi}{3}\right)\right) \end{aligned}$$

for $n = 0, 1, 2$, and the roots of the equation $z^3 + 64 = 0$ are given by

$$n = 0 \implies z = 4\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right),$$

$$n = 1 \implies z = 4\left(\cos\pi + \mathrm{i}\sin\pi\right),$$

$$n = 2 \implies z = 4\left(\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}\right).$$

(b) Dividing the area of the triangle into smaller three identical triangles, we get

$$\begin{aligned} \text{Area} &= 3\left[\frac{1}{2}ab\sin\left(\mathrm{B\hat{C}A}\right)\right] \qquad \left(\text{where } a = b = 4 \text{ and } \mathrm{B\hat{C}A} = \frac{2\pi}{3}\right) \\ &= 3 \cdot \frac{1}{2} \cdot 4 \cdot 4 \cdot \frac{\sqrt{3}}{2} \\ &= 12\sqrt{3} \end{aligned}$$

as required.

5

[Maximum mark: 12]

Consider the complex numbers $z_1 = 3\,\mathrm{cis}(120°)$ and $z_2 = 2 + 2\mathrm{i}$.

(a) Calculate $\frac{z_1}{z_2}$ giving your answer both in modulus-argument form and Cartesian form. [7]

(b) Use your results from part (a) to find the exact value of $\sin 15° \cdot \sin 45° \cdot \sin 75°$ giving your answer in the form $\frac{\sqrt{a}}{b}$ where $a, b \in \mathbb{Z}^+$. [5]

Solution

(a) In modulus-argument form, we have

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{3\,\mathrm{cis}(120^\circ)}{2\sqrt{2}\,\mathrm{cis}(45^\circ)} \\ &= \frac{3\sqrt{2}}{4}\,\mathrm{cis}(75^\circ).\end{aligned}$$

In Cartesian form, we have

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{\frac{1}{2}\left(-3+3\sqrt{3}\mathrm{i}\right)}{2+2\mathrm{i}} \\ &= \frac{\frac{1}{2}\left(-3+3\sqrt{3}\mathrm{i}\right)}{2+2\mathrm{i}} \cdot \frac{2-2\mathrm{i}}{2-2\mathrm{i}} \\ &= \frac{\left(-3+3\sqrt{3}\mathrm{i}\right)(1-\mathrm{i})}{(2+2\mathrm{i})(2-2\mathrm{i})} \\ &= \frac{-3+3\mathrm{i}+3\sqrt{3}\mathrm{i}+3\sqrt{3}}{4+4} \\ &= \frac{3(\sqrt{3}-1)}{8} + \frac{3(\sqrt{3}+1)}{8}\mathrm{i}.\end{aligned}$$

(b) Equating the two expressions for $\frac{z_1}{z_2}$ from part (a), we get

$$\begin{aligned}\cos 75^\circ &= \frac{4}{3\sqrt{2}} \cdot \frac{3(\sqrt{3}-1)}{8} \\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \\ &\& \\ \sin 75^\circ &= \frac{4}{3\sqrt{2}} \cdot \frac{3(\sqrt{3}+1)}{8} \\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}.\end{aligned}$$

Hence we have

$$\begin{aligned}\sin 15^\circ \cdot \sin 45^\circ \cdot \sin 75^\circ &= \cos 75^\circ \cdot \sin 45^\circ \cdot \sin 75^\circ \\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}+1}{2\sqrt{2}} \\ &= \frac{(3-1)\sqrt{2}}{16} \\ &= \frac{\sqrt{2}}{8}\end{aligned}$$

as required.

[Maximum mark: 17]

(a) Solve the equation $z^3 = 27, z \in \mathbb{C}$, giving your answer in the form $z = r(\cos\theta + \mathrm{i}\sin\theta)$ and in the form $z = a + b\mathrm{i}$ where $a, b \in \mathbb{R}$. [6]

(b) Consider the complex numbers $z_1 = -1 + \mathrm{i}$ and $z_2 = \dfrac{1}{\sqrt{2}}\left(\cos\left(\dfrac{\pi}{3}\right) + \mathrm{i}\sin\left(\dfrac{\pi}{3}\right)\right)$.

(i) Write z_1 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$.

(ii) Calculate $z_1 z_2$ and write in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{R}$.

(iii) Hence find the value of $\tan\left(\dfrac{\pi}{12}\right)$ in the form $c + d\sqrt{3}$ where $c, d \in \mathbb{Z}$.

(iv) Find the smallest $p \in \mathbb{Q}^+$ such that $(z_1 z_2)^p$ is a positive real number. [11]

Solution

(a) In modulus-argument form, we have $z^3 = 27(\cos 0 + \mathrm{i}\sin 0)$. By De Moivre's theorem, we get

$$\begin{aligned} z &= \sqrt[3]{27}(\cos 0 + \mathrm{i}\sin 0)^{\frac{1}{3}} \\ &= 3\left(\cos\left(\frac{0+2n\pi}{3}\right) + \mathrm{i}\sin\left(\frac{0+2n\pi}{3}\right)\right) \\ &= 3\left(\cos\left(\frac{2n\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2n\pi}{3}\right)\right) \end{aligned}$$

for $n = 0, 1, 2$. Hence the roots of the equation $z^3 = 27$ are given by

$$\begin{aligned} n = 0 &\implies z = 3(\cos 0 + \mathrm{i}\sin 0) = 3, \\ n = 1 &\implies z = 3\left(\cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right) = -\frac{3}{2} + \frac{3\sqrt{3}}{2}\mathrm{i}, \\ n = 2 &\implies z = 3\left(\cos\left(\frac{4\pi}{3}\right) + \mathrm{i}\sin\left(\frac{4\pi}{3}\right)\right) = -\frac{3}{2} - \frac{3\sqrt{3}}{2}\mathrm{i}. \end{aligned}$$

(b) (i) $z_1 = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right)$.

(ii)

$$\begin{aligned} z_1 z_2 &= (-1+\mathrm{i})\left(\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\mathrm{i}\right) \\ &= \frac{-\sqrt{2}-\sqrt{6}}{4} + \frac{\sqrt{2}-\sqrt{6}}{4}\mathrm{i}. \end{aligned}$$

(iii)

$$\begin{aligned} z_1 z_2 &= \sqrt{2}\,\mathrm{cis}\left(\frac{3\pi}{4}\right)\cdot\frac{1}{\sqrt{2}}\,\mathrm{cis}\left(\frac{\pi}{3}\right) \\ &= \mathrm{cis}\left(\frac{3\pi}{4} + \frac{\pi}{3}\right) \\ &= \mathrm{cis}\left(\frac{13\pi}{12}\right) \\ &= \cos\left(\frac{13\pi}{12}\right) + \mathrm{i}\sin\left(\frac{13\pi}{12}\right). \end{aligned}$$

By comparing above two expressions, we obtain

$$\cos\left(\frac{13\pi}{12}\right) = \frac{-\sqrt{2}-\sqrt{6}}{4} \quad \text{and} \quad \sin\left(\frac{13\pi}{12}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}.$$

Hence we get

$$\begin{aligned} \tan\left(\frac{\pi}{12}\right) &= \tan\left(\frac{13\pi}{12}\right) \\ &= \sin\left(\frac{13\pi}{12}\right) \div \cos\left(\frac{13\pi}{12}\right) \\ &= \frac{\sqrt{2}-\sqrt{6}}{4} \div \frac{-\sqrt{2}-\sqrt{6}}{4} \\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\cdot\frac{4}{-\sqrt{2}-\sqrt{6}} \\ &= \frac{\sqrt{2}-\sqrt{6}}{-\sqrt{2}-\sqrt{6}}\cdot\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}-\sqrt{6}} \\ &= \frac{8-4\sqrt{3}}{4} \\ &= 2-\sqrt{3}. \end{aligned}$$

(iv) By De Moivre's theorem, $(z_1 z_2)^p = \mathrm{cis}\left(\frac{13p\pi}{12}\right)$ is a positive real number when $p = \frac{24}{13}$.

[Maximum mark: 16]

(a) Find the roots of $z^{16} = 1$ which satisfy the condition $0 < \arg(z) < \frac{\pi}{2}$, expressing your answer in the form $re^{i\theta}$, where $r, \theta \in \mathbb{R}^+$. [5]

(b) Let S be the sum of the roots found in part (a).

(i) Show that $\mathrm{Re}(S) = \mathrm{Im}(S)$.

(ii) By writing $\frac{\pi}{8}$ as $\frac{1}{2} \cdot \frac{\pi}{4}$, find the value of $\cos\left(\frac{\pi}{8}\right)$ in the form $\frac{\sqrt{a+\sqrt{b}}}{c}$, where a, b and c are integers to be determined.

(iii) Hence, or otherwise, show that $S = \frac{1}{2}\left(\sqrt{2+\sqrt{2}} + \sqrt{2} + \sqrt{2-\sqrt{2}}\right)(1+i)$. [11]

Solution

(a) If we write $z = r(\cos\theta + i\sin\theta)$ and $1 = \cos 0 + i \sin 0$, we have

$$\begin{aligned} z^{16} &= 1 \\ \left[r(\cos\theta + i\sin\theta)\right]^{16} &= 1(\cos 0 + i\sin 0) \\ r^{16}(\cos 16\theta + i \sin 16\theta) &= 1(\cos 0 + i \sin 0) \qquad \text{[by De Moivre's theorem]} \end{aligned}$$

Hence we get $r = 1^{1/16} = 1$ and

$$\begin{aligned} \theta &= \frac{2n\pi}{16} \\ &= \frac{n\pi}{8} \qquad [n \in \mathbb{Z}] \end{aligned}$$

Since the roots must satisfy the condition $0 < \arg(z) < \pi/2$, we find $n = 1, 2, 3$.

Hence the roots of $z^{16} = 1$ whose argument is between 0 and $\pi/2$ are

$$\begin{aligned} z_1 &= e^{\frac{\pi i}{8}} \\ z_2 &= e^{\frac{\pi i}{4}} \\ z_3 &= e^{\frac{3\pi i}{8}} \end{aligned}$$

(b) (i) If we let $S = z_1 + z_2 + z_3$, we have

$$\begin{aligned} \mathrm{Re}(S) &= \cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{8}\right) \\ \mathrm{Im}(S) &= \sin\left(\frac{\pi}{8}\right) + \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{3\pi}{8}\right) \end{aligned}$$

Using the reflection law for cosine and sine, we get

$$\begin{aligned} \cos\left(\frac{\pi}{8}\right) &= \cos\left(\frac{\pi}{2} - \frac{3\pi}{8}\right) \\ &= \sin\left(\frac{3\pi}{8}\right) \end{aligned}$$

$$\begin{aligned} \cos\left(\frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) \\ &= \sin\left(\frac{\pi}{4}\right) \end{aligned}$$

$$\begin{aligned} \cos\left(\frac{3\pi}{8}\right) &= \cos\left(\frac{\pi}{2} - \frac{\pi}{8}\right) \\ &= \sin\left(\frac{\pi}{8}\right) \end{aligned}$$

Hence we deduce $\mathrm{Re}(S) = \mathrm{Im}(S)$.

(ii) Using the half-angle identity for cosine, we have

$$\begin{aligned} \cos\left(\frac{\pi}{8}\right) &= \cos\left(\frac{1}{2} \cdot \frac{\pi}{4}\right) \\ &= \sqrt{\frac{1 + \cos(\pi/4)}{2}} \\ &= \sqrt{\frac{1}{2}\left[1 + \frac{\sqrt{2}}{2}\right]} \\ &= \frac{\sqrt{2+\sqrt{2}}}{2} \end{aligned}$$

(iii) Similarly, using the half-angle identity for sine, we have

$$\begin{aligned} \cos\left(\frac{3\pi}{8}\right) &= \sin\left(\frac{\pi}{8}\right) \\ &= \sin\left(\frac{1}{2} \cdot \frac{\pi}{4}\right) \\ &= \sqrt{\frac{1 - \cos(\pi/4)}{2}} \\ &= \sqrt{\frac{1}{2}\left[1 - \frac{\sqrt{2}}{2}\right]} \\ &= \frac{\sqrt{2-\sqrt{2}}}{2} \end{aligned}$$

Hence we get

$$\begin{aligned} \mathrm{Re}(S) &= \cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{8}\right) \\ &= \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} \\ &= \frac{1}{2}\left(\sqrt{2+\sqrt{2}} + \sqrt{2} + \sqrt{2-\sqrt{2}}\right) \end{aligned}$$

Hence we obtain

$$\begin{aligned} S &= \mathrm{Re}(S) + i\,\mathrm{Im}(S) \\ &= \mathrm{Re}(S) + i\,\mathrm{Re}(S) \qquad [\text{since } \mathrm{Re}(S) = \mathrm{Im}(S)] \\ &= \mathrm{Re}(S)(1+i) \\ &= \frac{1}{2}\left(\sqrt{2+\sqrt{2}} + \sqrt{2} + \sqrt{2-\sqrt{2}}\right)(1+i) \end{aligned}$$

[Maximum mark: 19]

(a) (i) Expand $(\cos\theta + \mathrm{i}\sin\theta)^4$ by using the binomial theorem.

(ii) Hence use De Moivre's theorem to prove that $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$.

(iii) State a similar expression for $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$. [6]

Let $z = r(\cos\beta + \mathrm{i}\sin\beta)$, where β is measured in degrees, be the solution of $z^4 - \mathrm{i} = 0$ which has the smallest positive argument.

(b) Find the modulus and argument of z. [4]

(c) Use (a) (ii) and your answer from (b) to show that $8\cos^4\beta - 8\cos^2\beta + 1 = 0$. [4]

(d) Hence express $\cos 22.5°$ in the form $\dfrac{\sqrt{a+b\sqrt{c}}}{d}$ where $a, b, c, d \in \mathbb{Z}$. [5]

Solution

(a) (i) $(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta$.

(ii) By De Moivre's theorem, we also have

$$(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos 4\theta + \mathrm{i}\sin 4\theta.$$

Hence, by equating real parts of above two expressions, we get

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

(iii) By equating imaginary parts, we obtain

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.$$

(b)

$$\begin{aligned} z^4 - \mathrm{i} &= 0 \\ z^4 &= \mathrm{i} \\ z^4 &= \operatorname{cis}\left(\frac{\pi}{2}\right) \\ z &= \operatorname{cis}\left(\frac{\pi}{8} + \frac{n\pi}{2}\right), \quad n = 0, 1, 2, 3. \qquad \text{[by De Moivre's theorem]} \end{aligned}$$

Hence $r = 1$ and $\beta = \frac{\pi}{8} = 22.5°$.

(c) If we substitute $\theta = \beta$ into the identity form (a) (ii), then we get

$$\begin{aligned} \cos^4\beta - 6\cos^2\beta\sin^2\beta + \sin^4\beta &= \cos 4\beta \\ \cos^4\beta - 6\cos^2\beta(1-\cos^2\beta) + (1-\cos^2\beta)^2 &= \cos 90° \\ \cos^4\beta - 6\cos^2\beta(1-\cos^2\beta) + (1 - 2\cos^2\beta + \cos^4\beta) &= 0 \\ 8\cos^4\beta - 8\cos^2\beta + 1 &= 0. \end{aligned}$$

(d)

$$\begin{aligned} \cos^2\beta &= \frac{8 \pm \sqrt{64-32}}{16} \\ \cos^2\beta &= \frac{2 \pm \sqrt{2}}{4} \\ \cos\beta &= \pm\frac{\sqrt{2 \pm \sqrt{2}}}{2} \end{aligned}$$

Since $\cos 22.5° > \cos 45° = \frac{\sqrt{2}}{2} > 0$, both of the signs must be positive. Hence we get

$$\cos 22.5° = \frac{\sqrt{2+\sqrt{2}}}{2}$$

as required.

[Maximum mark: 22]

(a) Solve the equation $\sin(x + 90°) = 2\cos(x - 60°)$, $0° < x < 360°$. [5]

(b) Show that $\sin 15° + \cos 15° = \dfrac{\sqrt{6}}{2}$. [3]

(c) Let $z = 1 - \cos 4\theta - \mathrm{i}\sin 4\theta$, $z \in \mathbb{C}$, $0 < \theta < \dfrac{\pi}{2}$.

(i) Find the modulus and argument of z. Express each answer in its simplest form.

(ii) Hence find the fourth roots of z in modulus-argument form. [14]

Solution

(a) Using sum and difference formulas for sine and cosine, we get

$$\begin{aligned}
\sin(x+90^\circ) &= 2\cos(x-60^\circ)\\
\sin x\cos 90^\circ + \cos x \sin 90^\circ &= 2\cos x\cos 60^\circ + 2\sin x \sin 60^\circ\\
(\sin x)\cdot 0 + (\cos x)\cdot 1 &= (2\cos x)\cdot\frac{1}{2} + (2\sin x)\cdot\frac{\sqrt{3}}{2}\\
\cos x &= \cos x + \sqrt{3}\sin x\\
0 &= \sin x\\
x &= 180^\circ.
\end{aligned}$$

(b) Using the difference of angles $45^\circ - 30^\circ = 15^\circ$, we get

$$\begin{aligned}
\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\
&= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\\
&= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

and

$$\begin{aligned}
\cos 15^\circ &= \cos(45^\circ - 30^\circ)\\
&= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ\\
&= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\
&= \frac{\sqrt{6}+\sqrt{2}}{4}
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\sin 15^\circ + \cos 15^\circ &= \frac{\sqrt{6}-\sqrt{2}}{4} + \frac{\sqrt{6}+\sqrt{2}}{4}\\
&= \frac{\sqrt{6}}{2}
\end{aligned}$$

as required.

(c) (i) The modulus of z is

$$\begin{aligned}
|z| &= \sqrt{(1-\cos 4\theta)^2 + (\sin 4\theta)^2}\\
&= \sqrt{1 - 2\cos 4\theta + (\cos 4\theta)^2 + (\sin 4\theta)^2}\\
&= \sqrt{2 - 2\cos 4\theta}\\
&= \sqrt{2 - 2[1 - 2(\sin 2\theta)^2]}\\
&= \sqrt{4(\sin 2\theta)^2}\\
&= 2\sin 2\theta. \qquad [\text{since } 0 < 2\theta < \pi]
\end{aligned}$$

Letting $\arg z = \beta$, we have

$$\begin{aligned}
\tan\beta &= -\frac{\sin 4\theta}{1-\cos 4\theta}\\
&= -\frac{2\sin 2\theta\cos 2\theta}{2(\sin 2\theta)^2}\\
&= -\frac{\cos 2\theta}{\sin 2\theta}\\
&= -\cot 2\theta\\
&= -\tan\left(\frac{\pi}{2} - 2\theta\right)\\
&= \tan\left(2\theta - \frac{\pi}{2}\right)
\end{aligned}$$

Hence the argument of z is

$$\begin{aligned}
\arg z &= \beta\\
&= \arctan\left(\tan\left(2\theta - \frac{\pi}{2}\right)\right)\\
&= 2\theta - \frac{\pi}{2}
\end{aligned}$$

(ii) Applying De Moivre's theorem, the fourth roots of z are

$$(1 - \cos 4\theta - \mathrm{i}\sin 4\theta)^{\frac{1}{4}} = (2\sin 2\theta)^{\frac{1}{4}}\operatorname{cis}\left(\frac{\theta}{2} - \frac{\pi}{8} + \frac{n\pi}{2}\right)$$

for $n = 0, 1, 2, 3$.

[Maximum mark: 15]

Consider $w = \dfrac{z-1}{z+\mathrm{i}}$ where $z = x + \mathrm{i}y$ and $\mathrm{i} = \sqrt{-1}$.

(a) If $z = \mathrm{i}$,

(i) write w in the form $r\operatorname{cis}\theta$;

(ii) find the value of w^{14}. [5]

(b) Show that in general,

$$w = \frac{(x^2 - x + y^2 + y) + \mathrm{i}(y - x + 1)}{x^2 + (y+1)^2}$$ [4]

(c) Under what condition is $\mathbf{Re}(w) = 1$. [2]

(d) Under what condition is w:

(i) real;

(ii) purely imaginary. [2]

(e) Find the modulus of z given that $\arg(w) = \dfrac{\pi}{4}$. [2]

Solution

(a) (i) For $z = \mathrm{i}$, we have

$$\begin{aligned} w &= \frac{\mathrm{i}-1}{\mathrm{i}+\mathrm{i}} \\ &= \frac{\mathrm{i}-1}{2\mathrm{i}} \cdot \frac{\mathrm{i}}{\mathrm{i}} \\ &= \frac{-1-\mathrm{i}}{2\cdot(-1)} \\ &= \frac{1+\mathrm{i}}{2} \end{aligned}$$

Hence we get

$$\begin{aligned} |w| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \\ &= \sqrt{\frac{1}{4}+\frac{1}{4}} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

and

$$\begin{aligned} \arg w &= \theta \\ \tan\theta &= \frac{1/2}{1/2} = 1 \\ &\Longrightarrow \\ \arg w &= \frac{\pi}{4} \end{aligned}$$

so that

$$w = \frac{1}{\sqrt{2}}\operatorname{cis}\left(\frac{\pi}{4}\right).$$

(ii) We have

$$\begin{aligned} w^{14} &= \left(\frac{1}{\sqrt{2}}\right)^{14}\operatorname{cis}\left(\frac{14\pi}{4}\right) \\ &= \frac{1}{2^7}\operatorname{cis}\left(\frac{7\pi}{2}\right) \\ &= \frac{1}{128}(-\mathrm{i}) \\ &= -\frac{\mathrm{i}}{128}. \end{aligned}$$

(b) We have

$$\begin{aligned} w &= \frac{x+\mathrm{i}y-1}{x+\mathrm{i}y+\mathrm{i}} \\ &= \frac{(x-1)+\mathrm{i}y}{x+\mathrm{i}(y+1)} \cdot \frac{x-\mathrm{i}(y+1)}{x-\mathrm{i}(y+1)} \\ &= \frac{x(x-1)+y(y+1)+\mathrm{i}\big(xy-(x-1)(y+1)\big)}{x^2+(y+1)^2} \\ &= \frac{(x^2-x+y^2+y)+\mathrm{i}\big(\cancel{xy}-\cancel{xy}-x+y+1\big)}{x^2+(y+1)^2} \\ &= \frac{(x^2-x+y^2+y)+\mathrm{i}(y-x+1)}{x^2+(y+1)^2} \end{aligned}$$

as required.

(c) We have

$$\begin{aligned} \mathbf{Re}(w) &= 1 \\ \frac{x^2-x+y^2+y}{x^2+(y+1)^2} &= 1 \\ \cancel{x^2}-x+\cancel{y^2}+y &= \cancel{x^2}+\cancel{y^2}+2y+1 \\ y &= -x-1 \quad \text{for} \quad (x,y)\neq(0,-1). \end{aligned}$$

(d) (i) w is real if

$$\begin{aligned} y-x+1 &= 0 \\ y &= x-1 \quad \text{for} \quad (x,y)\neq(0,-1). \end{aligned}$$

(ii) w is purely imaginary if

$$x^2-x+y^2+y=0 \quad \text{and} \quad y-x+1\neq 0 \quad \text{for} \quad (x,y)\neq(0,-1).$$

(e) We have

$$\begin{aligned} \arg(w) &= \frac{\pi}{4} \\ 1 &= \frac{y-x+1}{x^2-x+y^2+y} \\ x^2-x+y^2+y &= y-x+1 \\ x^2+y^2 &= 1 \\ \sqrt{x^2+y^2} &= 1 \\ |z| &= 1. \end{aligned}$$

[Maximum mark: 21]

(a) Use de Moivre's theorem to find the value of $\left[\cos\left(\frac{\pi}{6}\right)+\mathrm{i}\sin\left(\frac{\pi}{6}\right)\right]^{12}$. [2]

(b) Use mathematical induction to prove that

$$(\cos\alpha - \mathrm{i}\sin\alpha)^n = \cos(n\alpha) - \mathrm{i}\sin(n\alpha) \text{ for } n\in\mathbb{Z}^+.$$ [6]

Let $w = \cos\alpha + \mathrm{i}\sin\alpha$.

(c) Find an expression in terms of α for $w^n - (w^*)^n$, $n\in\mathbb{Z}^+$ where w^* is the complex conjugate of w. [2]

(d) (i) Show that $ww^* = 1$.

(ii) Write down and simplify the binomial expansion of $(w-w^*)^3$ in terms of w and w^*.

(iii) Hence show that $\sin(3\alpha) = 3\sin\alpha - 4\sin^3\alpha$. [5]

(e) Hence solve $4\sin^3\alpha + (2\cos\alpha - 3)\sin\alpha = 0$ for $0\leq\alpha\leq\pi$. [6]

Solution

(a) Using de Moivre's theorem, we get

$$\begin{aligned}\left[\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right]^{12} &= \cos(2\pi) + \mathrm{i}\sin(2\pi) \\ &= 1.\end{aligned}$$

(b) Let $\mathrm{P}(n)$ be the following statement:

$$(\cos\alpha - \mathrm{i}\sin\alpha)^n = \cos(n\alpha) - \mathrm{i}\sin(n\alpha)$$

for $n \in \mathbb{Z}^+$. Using induction, we will prove that $\mathrm{P}(n)$ holds for all $n \in \mathbb{Z}^+$.

Base case. For $n = 1$, we have

$$\begin{aligned}(\cos\alpha - \mathrm{i}\sin\alpha)^1 &= \cos\alpha - \mathrm{i}\sin\alpha \\ &= \cos(1\cdot\alpha) - \mathrm{i}\sin(1\cdot\alpha).\end{aligned}$$

So, $\mathrm{P}(1)$ holds.

Inductive hypothesis. Assume that $\mathrm{P}(k)$ holds for some $k \geq 1$. In other words, assume that we have the following equality

$$(\cos\alpha - \mathrm{i}\sin\alpha)^k = \cos(k\alpha) - \mathrm{i}\sin(k\alpha) \qquad (\star)$$

for some $k \geq 1$.

Inductive step: To show that $\mathrm{P}(k+1)$ holds, we need to show that

$$(\cos\alpha - \mathrm{i}\sin\alpha)^{k+1} = \cos\left((k+1)\alpha\right) - \mathrm{i}\sin\left((k+1)\alpha\right) \qquad (\star\star)$$

If we start from the left hand side of the equality $(\star\star)$, then we obtain

$$\begin{aligned}(\cos\alpha - \mathrm{i}\sin\alpha)^{k+1} &= (\cos\alpha - \mathrm{i}\sin\alpha)^k\cos\alpha - (\cos\alpha - \mathrm{i}\sin\alpha)^k(\mathrm{i}\sin\alpha) \\ &= \big(\cos(k\alpha) - \mathrm{i}\sin(k\alpha)\big)\cos\alpha - \big(\cos(k\alpha) - \mathrm{i}\sin(k\alpha)\big)(\mathrm{i}\sin\alpha) \quad [\text{by } (\star)] \\ &= \cos(k\alpha)\cos\alpha - \mathrm{i}\sin(k\alpha)\cos\alpha - \mathrm{i}\cos(k\alpha)\sin\alpha - \sin(k\alpha)\sin\alpha \\ &= \cos(k\alpha)\cos\alpha - \sin(k\alpha)\sin\alpha - \mathrm{i}\big(\sin(k\alpha)\cos\alpha + \cos(k\alpha)\sin\alpha\big) \\ &= \cos\left((k+1)\alpha\right) - \mathrm{i}\sin\left((k+1)\alpha\right).\end{aligned}$$

Hence $\mathrm{P}(k+1)$ holds as well.

Conclusion. By mathematical induction, we can conclude that

$$(\cos\alpha - \mathrm{i}\sin\alpha)^n = \cos(n\alpha) - \mathrm{i}\sin(n\alpha)$$

holds for all $n \in \mathbb{Z}^+$.

(c) We have

$$\begin{aligned}w^n - (w^*)^n &= (\cos\alpha + \mathrm{i}\sin\alpha)^n - (\cos\alpha - \mathrm{i}\sin\alpha)^n \\ &= \big(\cancel{\cos(n\alpha)} + \mathrm{i}\sin(n\alpha)\big) - \big(\cancel{\cos(n\alpha)} - \mathrm{i}\sin(n\alpha)\big) \quad [\text{by part (b)}] \\ &= 2\mathrm{i}\sin(n\alpha).\end{aligned}$$

(d) (i) We have

$$\begin{aligned}ww^* &= (\cos\alpha + \mathrm{i}\sin\alpha)(\cos\alpha - \mathrm{i}\sin\alpha) \\ &= \cos^2\alpha - \cancel{\mathrm{i}\cos\alpha\sin\alpha} + \cancel{\mathrm{i}\sin\alpha\cos\alpha} + \sin^2\alpha \\ &= \cos^2\alpha + \sin^2\alpha \\ &= 1\end{aligned}$$

as required.

(ii) We have

$$\begin{aligned}(w - w^*)^3 &= w^3 - 3w^2w^* + 3w(w^*)^2 - (w^*)^3 \\ &= w^3 - 3w + 3w^* - (w^*)^3. \quad [\text{by part (d) (i)}]\end{aligned}$$

(iii) Hence we get

$$\begin{aligned}(w - w^*)^3 &= w^3 - (w^*)^3 - 3(w - w^*) \\ (2\mathrm{i}\sin\alpha)^3 &= 2\mathrm{i}\sin(3\alpha) - 3(2\mathrm{i}\sin\alpha) \quad [\text{by part (c)}] \\ -8\mathrm{i}\sin^3\alpha &= 2\mathrm{i}\sin(3\alpha) - 6\mathrm{i}\sin\alpha \quad [\text{divide by } 2\mathrm{i}] \\ -4\sin^3\alpha &= \sin(3\alpha) - 3\sin\alpha \\ \sin(3\alpha) &= 3\sin\alpha - 4\sin^3\alpha\end{aligned}$$

as required.

(e) We have

$$\begin{aligned}4\sin^3\alpha + (2\cos\alpha - 3)\sin\alpha &= 0 \\ 2\sin\alpha\cos\alpha &= 3\sin\alpha - 4\sin^3\alpha \\ \sin(2\alpha) &= \sin(3\alpha). \quad [\text{by part (d) (iii)}]\end{aligned}$$

Hence we get

$$\alpha = 0$$

$$\begin{aligned}3\alpha &= \pi - 2\alpha \implies \alpha = \frac{\pi}{5} \\ 3\alpha &= 3\pi - 2\alpha \implies \alpha = \frac{3\pi}{5} \\ 3\alpha &= 5\pi - 2\alpha \implies \alpha = \pi\end{aligned}$$

Q5. [2013/Prelim/MJC/II/1]

A graphing calculator is **not** to be used in answering this question.

One root of the equation $x^4 + ax^3 + 5x^2 - x - 10 = 0$, where a is real, is $x = 1 + 2\text{i}$. Find the value of a and the other roots. **[5]**

Hence find the x-coordinates of the point(s) of intersection between the graphs of $y = 5\left(x^2 - 2x^4\right)$ and $y = x^3 + 3x - 1$. **[3]**

<u>Solution</u>

Let f(x) = $x^4 + ax^3 + 5x^2 - x - 10$.

Since the coefficients of f(x) are real, and 1 + 2i is a root of f(x) = 0, therefore 1 − 2i is also a root.

$$\begin{aligned} \text{f}(x) &= \left(x - (1+2\text{i})\right)\left(x - (1-2\text{i})\right)\left(x^2 + bx + c\right) \\ &= \left((x-1) - 2\text{i}\right)\left((x-1) + 2\text{i}\right)\left(x^2 + bx + c\right) \\ &= \left((x-1)^2 - (2\text{i})^2\right)\left(x^2 + bx + c\right) \\ &= \left(x^2 - 2x + 5\right)\left(x^2 + bx + c\right) \end{aligned}$$

Comparing coefficients of

constant: $c = -2$

x: $5b - 2c = -1 \Rightarrow b = -1$

x^3: $-2 - 1 = a \Rightarrow a = -3$

$\therefore \text{f}(x) = \left(x^2 - 2x + 5\right)\left(x^2 - x - 2\right) = \left(x^2 - 2x + 5\right)(x-2)(x+1)$

The other roots are 1 − 2i, −1 and 2.

$$5\left(x^2 - 2x^4\right) = x^3 + 3x - 1$$

$$-10x^4 - x^3 + 5x^2 - 3x + 1 = 0$$

$$\frac{1}{x^4} - \frac{3}{x^3} + \frac{5}{x^2} - \frac{1}{x} - 10 = 0$$

Replace x by $\frac{1}{x}$,

$\frac{1}{x} = -1$ or $\frac{1}{x} = 2$

$x = -1$ or $x = \frac{1}{2}$

Q8. **[2013/Prelim/TJC/I/5]**

A graphic calculator is **not** to be used in answering this question

Two complex numbers p and q are given by $p = 1 - \mathrm{i}$ and $q = -1 + \sqrt{3}\mathrm{i}$, and $z = \frac{p}{q}$.

(i) Express z in the form $x + y\mathrm{i}$, where x and y are exact real values to be determined. **[2]**

(ii) By considering the moduli and arguments of p and q, find the exact values of $|z|$ and arg z, where $-\pi < \arg z \le \pi$. **[4]**

(iii) Hence, show that $\sin\left(\frac{11}{12}\pi\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$. **[3]**

Solution

(i) $z = \frac{p}{q} = \left(\frac{1-\mathrm{i}}{-1+\sqrt{3}\mathrm{i}}\right)\left(\frac{-1-\sqrt{3}\mathrm{i}}{-1-\sqrt{3}\mathrm{i}}\right) = \frac{-1-\sqrt{3}\mathrm{i}+\mathrm{i}-\sqrt{3}}{4} = \frac{1}{4}\left(-1-\sqrt{3}\right) + \frac{1}{4}\left(1-\sqrt{3}\right)\mathrm{i}$

(ii) $|p| = \sqrt{1^2+1^2} = \sqrt{2}$, $\arg p = -\tan^{-1} 1 = -\frac{\pi}{4}$.

$|q| = \sqrt{1^2+\left(\sqrt{3}\right)^2} = 2$, $\arg q = \pi - \tan^{-1}\sqrt{3} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

$\Rightarrow |z| = \frac{|p|}{|q|} = \frac{\sqrt{2}}{2}$, $\arg z = \arg p - \arg q = -\frac{\pi}{4} - \frac{2\pi}{3} = -\frac{11}{12}\pi$

(iii) From part **(i)** and **(ii)**,

$\frac{1}{4}\left(-1-\sqrt{3}\right) + \frac{1}{4}\left(1-\sqrt{3}\right)\mathrm{i} = \frac{\sqrt{2}}{2}\left(\cos\left(-\frac{11\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{11\pi}{12}\right)\right)$

Comparing the imaginary parts, $\frac{\sqrt{2}}{2}\sin\left(-\frac{11\pi}{12}\right) = \frac{1}{4}\left(1-\sqrt{3}\right)$

Since $\sin\left(-\frac{11\pi}{12}\right) = -\sin\left(\frac{11\pi}{12}\right)$,

we get $\sin\left(\frac{11\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$