1

Let's first find the intercepts for this function.

The y-intercept is the point $(0, f(0)) = (0, 2)$.

For the x-intercepts we set the numerator equal to zero and solve. However, in this case the numerator is a constant (-4 specifically) and so can't ever be zero. Therefore, this function will have no x-intercepts.

Hide Step 2 ▾

We can find any vertical asymptotes be setting the denominator equal to zero and solving. Doing that for this function gives,

$$x - 2 = 0 \quad \rightarrow \quad x = 2$$

So, we'll have a vertical asymptote at $x = 2$.

Hide Step 3 ▾

For this equation the largest exponent of x in the numerator is zero since the numerator is a constant. The largest exponent of x in the denominator is 1, which is larger than the largest exponent in the numerator, and so the x-axis will be the horizontal asymptote.

Hide Step 4 ▾

From Step 2 we saw we only have one vertical asymptote and so we only have two regions to our graph : $x < 2$ and $x > 2$.

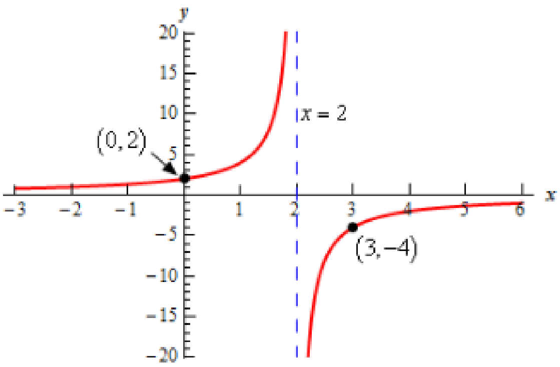
We'll need a point in each region to determine if it will be above or below the horizontal asymptote. Here are a couple of function evaluations for the points.

$$\begin{aligned} f(0) &= 2 \quad &\rightarrow \quad &(0, 2) \\ f(3) &= -4 \quad &\rightarrow \quad &(3, -4) \end{aligned}$$

Note that the first evaluation didn't really need to be done since it was just the y-intercept which we had already found in the first step. It was included here mostly for the sake of completeness.

Hide Step 5 ▾

Here is a sketch of the function with the points found above. The vertical asymptote is indicated with a blue dashed line and recall that the horizontal asymptote is just the x-axis.



2

Let's first find the intercepts for this function.

The y-intercept is the point $(0, f(0)) = (0, 6)$.

For the x-intercepts we set the numerator equal to zero and solve. Doing that for this problem gives,

$$6 - 2x = 0 \quad \rightarrow \quad x = 3$$

So, the only x-intercept for this problem is $(3, 0)$.

Hide Step 2 ▾

We can find any vertical asymptotes be setting the denominator equal to zero and solving. Doing that for this function gives,

$$1 - x = 0 \quad \rightarrow \quad x = 1$$

So, we'll have a vertical asymptote at $x = 1$.

Hide Step 3 ▾

For this equation the largest exponent of x in both the numerator and denominator is 1. Therefore, the horizontal asymptote for this problem is then the coefficient of the x in the numerator divided by the coefficient of the x in the denominator. Or,

$$y = \frac{-2}{-1} = 2$$

Hide Step 4 ▾

From Step 2 we saw we only have one vertical asymptote and so we only have two regions to our graph : $x < 1$ and $x > 1$.

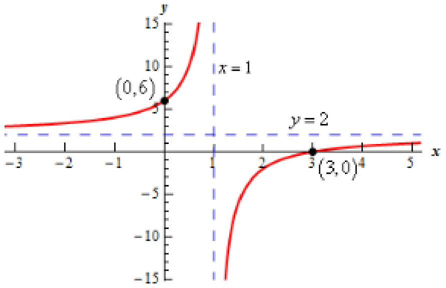
We'll need a point in each region to determine if it will be above or below the horizontal asymptote. Here are a couple of function evaluations for the points.

$$f(0) = 6 \quad \rightarrow \quad (0, 6)$$
$$f(3) = 0 \quad \rightarrow \quad (3, 0)$$

Note that both of these are the intercepts we found in the first step. In this case they both just happened to be on either side of the vertical asymptote and so we could use these two points here.

Hide Step 5 ▾

Here is a sketch of the function with the points found above. The vertical and horizontal asymptotes are indicated with blue dashed lines.



3

Let's first find the intercepts for this function.

The y-intercept is the point $(0, f(0)) = \left(0, -\frac{4}{3}\right)$.

For the x-intercepts we set the numerator equal to zero and solve. However, in this case the numerator is a constant (8 specifically) and so can't ever be zero. Therefore, this function will have no x-intercepts.

Hide Step 2 ▾

We can find any vertical asymptotes be setting the denominator equal to zero and solving. Doing that for this function gives,

$$x^2 + x - 6 = (x+3)(x-2) = 0 \quad \rightarrow \quad x = -3, \quad x = 2$$

So, we'll have two vertical asymptotes at $x = -3$ and $x = 2$.

Hide Step 3 ▾

For this equation the largest exponent of x in the numerator is zero since the numerator is a constant. The largest exponent of x in the denominator is 2, which is larger than the largest exponent in the numerator, and so the x-axis will be the horizontal asymptote.

Hide Step 4 ▾

From Step 2 we saw we only have two vertical asymptotes and so we have three regions to our graph : $x < -3$, $-3 < x < 2$ and $x > 2$.

We'll need a point in each region to determine if it will be above or below the horizontal asymptote.

Also, as we discussed in the notes, there are a couple of possible different behaviors in the middle region. To determine just what the behavior is we need to get a couple of points in this region. The best idea for points in the middle region is check a couple of points close to the vertical asymptotes we know if the edge is going to be above or below the horizontal asymptote.

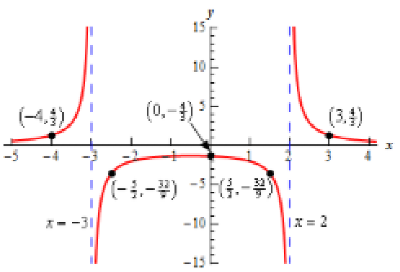
Here are some function evaluations for the points.

$$\begin{aligned} f(-4) &= \frac{4}{3} & &\rightarrow & &\left(-4, \frac{4}{3}\right) \\ f\left(-\frac{5}{2}\right) &= -\frac{32}{9} & &\rightarrow & &\left(-\frac{5}{2}, -\frac{32}{9}\right) \\ f\left(\frac{3}{2}\right) &= -\frac{32}{9} & &\rightarrow & &\left(\frac{3}{2}, -\frac{32}{9}\right) \\ f(3) &= \frac{4}{3} & &\rightarrow & &\left(3, \frac{4}{3}\right) \end{aligned}$$

From the second and third points we see that the curve in the middle region should be below the horizontal asymptote (x-axis for this problem) at both edges and so the curve will be completely below the horizontal asymptote in this whole region.

Hide Step 5 ▾

Here is a sketch of the function with the points found above. The vertical asymptote is indicated with a blue dashed line and recall that the horizontal asymptote is just the x-axis.



4

Let's first find the intercepts for this function.

The y-intercept is the point $(0, f(0)) = \left(0, \frac{9}{2}\right)$.

For the x-intercepts we set the numerator equal to zero and solve. Doing that for this problem gives,

$$4x^2 - 36 = 0 \qquad \to \qquad x^2 = 9 \qquad \to \qquad x = \pm 3$$

So, the two x-intercepts for this problem are $(-3, 0)$ and $(3, 0)$.

Hide Step 2 ▾

We can find any vertical asymptotes be setting the denominator equal to zero and solving. Doing that for this function gives,

$$x^2 - 2x - 8 = (x+2)(x-4) = 0 \qquad \to \qquad x = -2, \quad x = 4$$

So, we'll have two vertical asymptotes at $x = -2$ and $x = 4$.

Hide Step 3 ▾

For this equation the largest exponent of x in both the numerator and denominator is 2. Therefore, the horizontal asymptote for this problem is then the coefficient of the x^2 in the numerator divided by the coefficient of the x^2 in the denominator. Or,

$$y = \frac{4}{1} = 4$$

Hide Step 4 ▾

From Step 2 we saw we only have two vertical asymptotes and so we have three regions to our graph : $x < -2$, $-2 < x < 4$ and $x > 4$.

We'll need a point in each region to determine if it will be above or below the horizontal asymptote.

Also, as we discussed in the notes, there are a couple of possible different behaviors in the middle region. To determine just what the behavior is we need to get a couple of points in this region. The best idea for points in the middle region is check a couple of points close to the vertical asymptotes we know if the edge is going to be above or below the horizontal asymptote.

Here are some function evaluations for the points.

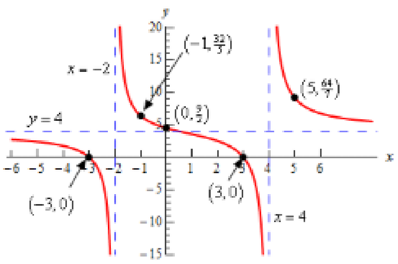
$$\begin{aligned} f(-3) &= 0 & &\to & &(-3, 0) \\ f(-1) &= \frac{32}{5} & &\to & &\left(-1, \frac{32}{5}\right) \\ f(3) &= 0 & &\to & &(3, 0) \\ f(5) &= \frac{64}{7} & &\to & &\left(5, \frac{64}{7}\right) \end{aligned}$$

From the second and third points we see that the curve in the middle region should be below the horizontal asymptote at the right edge and above the horizontal asymptote at the left edge and so the curve will cross the horizontal asymptote in this region.

Also note that we used the two x-intercepts here because they worked out to be good choices for points to use.

Hide Step 5 ▾

Here is a sketch of the function with the points found above. The vertical and horizontal asymptotes are indicated with blue dashed lines.



5-c

6-a

7-d

8-b

9

To fulfill these conditions, we can construct a rational function. First, we need to examine what these conditions mean for that type of a function.

a. When the horizontal asymptote of a rational function is a nonzero constant, that means that the degree of the numerator is equal to the degree of the denominator. The value of this constant is the ratio of the leading coefficients. Thus, the numerator and denominator of this rational function must have the same degree, and the ratio of their leading coefficients must be 2.

b. Since the discontinuity at 2 isn't a vertical asymptote, the numerator and denominator must both equal zero when x equals 2. Thus, the term $x-2$ must be present in both.

c. Since there aren't any other discontinuities or asymptotes, the denominator cannot be zero anywhere else. The easiest way to ensure this is to not add any other expressions besides what we've already defined needs to be there.

The most simple rational function that satisfies these requirements is:

$$\begin{aligned} y &= \frac{2(x-2)}{x-2} \\ &= \frac{2x-4}{x-2} \end{aligned}$$

The degree of the numerator and denominator are the same, and the ratio of the leading coefficients is 2, so the horizontal asymptote of $y=2$ exists. The numerator and denominator are both zero at $x=2$, so this is a discontinuity but not a vertical asymptote. The denominator is not zero anywhere else, so there are no other discontinuities or asymptotes. Therefore, this function meets our requirements.

10

The graph given has two vertical asymptotes, at $x = -3$ and at $x = 6$, and a horizontal asymptote, at $y = -3$. Thus, we need to construct a function that would have these three asymptotes. Let's begin with the two vertical asymptotes.

A function may have a vertical asymptote when the denominator equals zero. Thus, we need to construct a function where the denominator equals zero for the two given values of x above. The factored polynomial below does just that.

$$(x + 3)(x - 6)$$

Next, let's ensure that a horizontal asymptote exists where we want it. Since the horizontal asymptote is a nonzero constant, we know that the numerator and denominator have to have the same degree. Otherwise, the horizontal asymptote would be zero or non-existent. The ratio of the leading coefficient of the numerator and the leading coefficient of the denominator must equal the value of this asymptote. However, since making this ratio negative would flip the graph of the function across the x-axis, we need to change our method for ensuring this asymptote slightly. Instead, we will use a ratio of the leading coefficients that equal 3, and then we can shift the graph accordingly by subtracting 6. This moves the graph down six units, which moves the asymptote from positive to negative 3.

Since the denominator is a quadratic expression and has a leading coefficient of 1, we can put this information together to define the following function.

$$f(x) = \frac{3x^2}{(x+3)(x-6)} - 6$$

Lastly, we need to ensure that the vertical asymptotes still exist and equal the values intended. The only way that these asymptotes would not occur as we defined them would be if any term in the denominator cancels with any term in the numerator. Since this is not possible with how we constructed this function, we have indeed found a possible function that describes the behavior of this graph.

11

Explanantion

Asymptotes:

If $\lim_{x \to \infty} f(x) = L$ OR $\lim_{x \to -\infty} f(x) = L$, then *f* has a horizontal asymptote at *y*=*L*.

With rational functions, we can follow these rules:

(1) If the degree of the function in the numerator is bigger than the degree of the function in the denominator, then the function will not have any horizontal asymptotes. (This is because the leading term in the numerator will dominate as *x* gets really large, and thus the limit as described above will be positive or negative infinity.)

(2) if the degree of the function in the denominator is bigger than the degree of the function in the numerator, then the function will have a horizontal asymptote at *y*=0. (This is because the leading term in the denominator will dominate as *x* gets really large, and thus the limit described will be zero.

(3) If the degree of the function in the numerator is equal to the degree of the function in the denominator, then you divide the leading coefficient of the top by the leading coefficient of the bottom to get *a*, and *y*=*a* is the horizontal asymptote. (This is because the leading terms of the numerator and the denominator will dominate the function, and the x^n will cancel out and leave you with just the coefficients.

Any value *a* for which $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$ is a vertical asymptote of the function.

Rational functions are a special case of this rule: Suppose $f(x) = \frac{p(x)}{q(x)}$.

If $q(a) = 0$, but $p(a) \neq 0$, then we say that *x*=*a* is a vertical asymptote for the function.

Answer and Explanation:

First, we will factor this function. This will help us in many pieces of this problem.

$$f(x) = \frac{2x^2 - 8}{x^2 - 5x - 6} = \frac{2(x^2 - 4)}{(x-6)(x+1)} = \frac{2(x-2)(x+2)}{(x-6)(x+1)}.$$

(a) Now, let us begin with the domain. We cannot divide by zero, so we need to set the denominator equal to zero and solve. This will give us the values of *x* that need to be excluded from the domain.

$$(x-6)(x+1) = 0 \text{ if and only if } x = 6 \text{ and } x = -1.$$

We can write the domain of the function as $(-\infty, -1) \cup (-1, 6) \cup (6, \infty)$.

(b) Now let us consider the intercepts.

To find the x-intercepts, we need to find out what values of *x* make the function equal to zero. We need to set the numerator equal to zero and solve.

$$2(x-2)(x+2) = 0 \text{ if and only if } x = 2 \text{ and } x = -2.$$

Thus, we can conclude $(2, 0)$ and $(-2, 0)$ are the x-intercepts for this function.

Now to find the y-intercepts, we need to find the value of $f(0)$

Let's evaluate:

$$f(0) = \frac{2(-2)(2)}{(-6)(1)} = \frac{-8}{-6} = \frac{4}{3}.$$

Thus, the y-intercept is $\left(0, \frac{4}{3}\right)$.

(c) Lastly, we will find the asymptotes.

To find the vertical asymptotes, we set the denominator equal to zero. Note that none of these values simultaneously make the numerator zero, so both solutions are in fact asymptotes.

$$x = 6 \text{ and } x = -1 \text{ are the vertical asymptototes.}$$

Next, to find the horizontal asymptote, we note that the degree of the numerator is 2 and the degree of the denominator is 2. Thus, the horizontal asymptote will occur at the value of the ratio of the leading coefficient of the numerator divided by the leading coefficient of the denominator.

$y = \frac{2}{1}$, or $y = 2$ is the horizontal asymptote.

Rational Functions

A rational function is a function with the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials, and $Q(x) \neq 0$. The domain of a rational function is the set of all possible real number values of x except the x-values that would make the denominator equal to zero. Lastly, an asymptote is a line that approaches the graph of a rational function but never actually meets it. There are two types asymptote: vertical and horizontal asymptote, and the ways to solve them are different.

Answer and Explanation:

a) To identify the domain of the given function, we must solve the roots of the denominator to know the restricted values of x:

$$x^2 - 16 = 0 \qquad \text{[Equate the denominator to zero]}$$
$$x^2 = 16 \qquad \text{[Isolate the variable from the constant]}$$
$$x = \pm 4 \qquad \text{[Square root both sides]}$$

Therefore, the domain of the function $g(x) = \frac{x^2 - x - 6}{x^2 - 16}$ is all real numbers except ± 4.

b) To solve for the x-intercept, we must let $g(x) = 0$, letting the numerator be equal to zero and we solve for the values of x:

$$\frac{x^2 - x - 6}{x^2 - 16} = 0 \qquad \text{[Equate the function to zero]}$$
$$x^2 - x - 6 = 0 \qquad \text{[Equate the numerator to zero]}$$
$$(x - 3)(x + 2) = 0 \qquad \text{[Factor the polynomial]}$$
$$\therefore x = 3, -2 \qquad \text{[Equate each factor to zero and get the roots]}$$

These values of x will become the x-coordinate of the x-intercepts. Therefore, the x-intercepts of this function are $(0, 3)$ and $(0, -2)$.

To solve for the y-intercepts, we need to let $x = 0$ then solve for y:

$$y = \frac{(0)^2 - (0) - 6}{(0)^2 - 16} \qquad \text{[Let x=0]}$$
$$y = \frac{-6}{-16} \qquad \text{[Solve for y]}$$
$$\therefore y = \frac{3}{8} \qquad \text{[Simplify]}$$

This value of y will become the y-coordinate of the y-intercept. Therefore, the y-intercepts of this function is $(\frac{3}{8}, 0)$

c) If we are given a rational function $f(x) = \frac{P(x)}{Q(x)}$, its vertical asymptote are found at the zeros of $Q(x)$. Based from our solution in (a), the zeros of the denominator of our function are $x = \pm 4$. That means, the function has vertical asymptotes at $x = 4$ and $x = -4$.

On the other hand, the horizontal asymptotes of a rational function can be deduced from the degrees of the polynomials $P(x)$ and $Q(x)$:

i. If the degree of $P(x)$ is less than the degree of $Q(x)$, the function has a horizontal asymptote at $y = 0$.

ii. If the degree of $P(x)$ is more than the degree of $Q(x)$, the function has no horizontal asymptote.

iii. If the degree of $P(x)$ is equal to the degree of $Q(x)$, the function has a horizontal asymptote at $y = \frac{A}{B}$, where A and B are the leading coefficients of $P(x)$ and $Q(x)$, respectively.

Notice that from the given function, the degrees of both the numerator and denominator are equal to 2, which means that it follows the third rule for horizontal asymptote. Both their leading coefficients are also equal to 1. This means that the function has a horizontal asymptote at $y = 1$.

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It so happens that this function can be simplified as:

$$\begin{aligned} y &= \frac{x^2 - x - 2}{x - 2} \\ &= \frac{(x-2)(x+1)}{x-2} \\ &= \frac{(x-2)(x+1)}{(x-2)} \\ &= x + 1 \end{aligned}$$

Then the full answer is:

domain: $x \neq 2$

vertical asymptotes: none

horizontal asymptote: none

slant asymptote: $y = x + 1$